Maxwell's Equations and Their Consequences

Karl M. Westerberg (2025)

Note: This document is a work in progress

Contents

1 Basic Principles

On occasion throughout this text, we will use the following short-hand notation:

$$\partial_i \psi = \frac{\partial \psi}{\partial x_i}$$

for $i = x$, y, or z and ψ is any scalar (or vector) field. If ψ is also time-dependent, we may use

$$\partial_t \psi = \frac{\partial \psi}{\partial t}$$

for time derivatives, as well as

$$\partial_\mu \psi = \begin{cases} \partial_i \psi & \mu = i = x,\ y, \text{ or } z \\ \partial_t \psi & \mu = t \end{cases}$$

for general space-time derivatives. Note that $(\vec{\nabla}\psi)_i = \partial_i \psi$ for any scalar field.

1.1 Maxwell's equations

Integral form:

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int_\Omega \rho \, dV \tag{1.1}$$

$$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_\Sigma \vec{B} \cdot d\vec{A} \tag{1.2}$$

$$\oint_{\partial\Omega} \vec{B} \cdot d\vec{A} = 0 \tag{1.3}$$

$$\oint_{\partial\Sigma} \vec{B} \cdot d\vec{s} = \mu_0 \int_\Sigma \vec{J} \cdot d\vec{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_\Sigma \vec{E} \cdot d\vec{A} \tag{1.4}$$

Differential form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1.5}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.6}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.7}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}) \tag{1.8}$$

Lorentz force law:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{1.9}$$

$$\vec{F} = \int_\Omega (\rho \vec{E} + \vec{J} \times \vec{B}) \, dV \tag{1.10}$$

Rate of work done:

$$P = \vec{F} \cdot \vec{v} = q\vec{v} \cdot \vec{E} \tag{1.11}$$

$$P = \int_\Omega \vec{J} \cdot \vec{E}\, dV \tag{1.12}$$

1.2 Plane waves

Source-free Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{1.13}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.14}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.15}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{1.16}$$

Plane waves:

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_m \cos(\vec{k} \cdot \vec{r} - \omega t) \\ \vec{B}(\vec{r}, t) &= \vec{B}_m \cos(\vec{k} \cdot \vec{r} - \omega t) \end{aligned}$$

$$\partial_\mu \vec{E} = \vec{E}_m(-\sin(\vec{k} \cdot \vec{r} - \omega t)\partial_\mu(\vec{k} \cdot \vec{r} - \omega t)$$

$$\partial_\mu(\vec{k} \cdot \vec{r} - \omega t) = \begin{cases} k_i & \mu = i = x,\ y, \text{ or } z \\ -\omega & \mu = t \end{cases}$$

Similar results for $\vec{B}$.
Thus, taking the derivatives of $\vec{E}$ and $\vec{B}$ amounts to the following substitutions:

$$\cos \to -\sin \qquad \vec{\nabla} \to \vec{k} \qquad \partial/\partial t \to -\omega$$

Plugging into the source-free Maxwell's equations and dividing out $-\sin(\vec{k} \cdot \vec{r} - \omega t)$ yields:

$$\begin{aligned} \vec{k} \cdot \vec{E}_m &= 0 \\ \vec{k} \times \vec{E}_m &= -(-\omega \vec{B}_m) \\ \vec{k} \cdot \vec{B}_m &= 0 \\ \vec{k} \times \vec{B}_m &= \mu_0 \epsilon_0 (-\omega) \vec{E}_m \end{aligned}$$

According to these equations, $\vec{E}_m$, $\vec{B}_m$, and $\vec{k}$ are all mutually perpendicular to each other, with $\vec{k} \times \vec{E}_m$ positive parallel to $\vec{B}_m$. Thus, if $\vec{k}$ is in the $+x$ direction and $\vec{E}_m$ is in the $+y$ direction, then $\vec{B}_m$ will point in the $+z$ direction.

If we take the magnitude of the second and fourth equations, we learn that

$$E_m = cB_m \qquad B_m = c\mu_0\epsilon_0 E_m$$

where $c = \omega/k$ is the speed of the wave. Combining these two equations yields $E_m = cB_m$ and $c = 1/\sqrt{\mu_0\epsilon_0}$. Plugging in values yields $c = 3.00 \times 10^8$ m/s, which is the accepted value for the speed of light. Evidently, Maxwell set out to unify electricity and magnetism with his equations and ended up unifying electricity, magnetism, and optics!

2 Conservation Laws

2.1 Conservation of charge

Charge distribution given by ρ. $\vec{J}$ describes flow... $\vec{J} \cdot d\vec{A}$ is rate of charge flow across surface in direction of $d\vec{A}$. Current through surface is given by $\int_\Sigma \vec{J} \cdot d\vec{A}$. If the charge distribution moves rigidly with velocity $\vec{v}$, then $\vec{J} = \rho\vec{v}$.

Charge is a conserved quantity. This is often stated in terms of a global conservation principle, namely that the total charge of the universe never changes. Mathematically, this can be stated as

$$\frac{d}{dt}\int \rho(\vec{r},t)\, dV = 0 \tag{2.1}$$

where the integral is taken over the entire universe.

This global charge conservation principle, by its very nature, only makes claims about the total charge in the universe. It does not rule out the possibility that charge can decrease in one region of the universe and simultaneously increase in another region of the universe. Of course, such "teleportation processes" can be ruled out by relativity (it violates both the speed-of-light upper bound as well as relativity of simultaneity).

In fact, we can assert a stronger version of charge conservation:

$$\frac{d}{dt}\int_\Omega \rho(\vec{r},t)\, dV = -\oint_{\partial\Omega} \vec{J}(\vec{r},t)\cdot d\vec{A} \tag{2.2}$$

This law states that the rate of change of the amount of charge in any particular region of space must be equal to minus the net rate of charge flow out of that region through its boundary. In other words, charge within a given region can only change if it flows in or out of that region. Teleportation processes are strictly forbidden.

If we reduce the region to a point and divide out the volume (with the help of the divergence theorem), we can express local charge conservation at a point:

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} \tag{2.3}$$

This statement must be true at all points in space, as well as for all time.

Eq. 2.3 (as well as its integral form, eq. 2.2) is known as an "equation of continuity", and is a statement of *local conservation of charge*. It is sometimes written equivalently as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Local charge conservation is built into Maxwell's equations. Indeed, if we take the divergence of the Ampere-Maxwell Law (eq. 1.8), we get

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}))$$

The divergence of a curl is zero. If we divide out μ_0 and then plug in Gauss' Law (eq. 1.5), we find

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\rho / \epsilon_0) \\ &= \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \end{aligned}$$

Note the importance of the "displacement current" term ($\mu_0 \epsilon_0 \partial E / \partial t$) in eq. 1.8. If that term is omitted from the Ampere-Maxwell Law (which would revert that law back to Ampere's Law), then we would simply get $\vec{\nabla} \cdot \vec{J} = 0$. This would only be true if the charge density remains constant at all points in space. If one considers steady currents flowing around complete loops, then this statement is in fact valid, and Ampere's Law works perfectly fine. However, if charge is accumulating anywhere (e.g., charging / discharging capacitor circuits), then the Maxwell correction term becomes necessary.

2.2 Conservation of energy

Energy is also a conserved quantity. It is also the case that electric and magnetic fields contain energy. In fact, the electromagnetic energy density is given by

$$u = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \frac{1}{\mu_0} B^2 \tag{2.4}$$

with an associated energy current density given by

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \tag{2.5}$$

The vector $\vec{S}$ is sometimes known as the Poynting vector. Rate of flow of energy across $d\vec{A}$ is given by $\vec{S} \cdot d\vec{A}$. Rate of flow across general surface is $\int_\Sigma \vec{S} \cdot d\vec{A}$.

Since energy is conserved, it is tempting to assert the following continuity equation:

$$\frac{d}{dt} \int_\Omega u \, dV = - \oint_{\partial \Omega} \vec{S} \cdot d\vec{A} \qquad ??$$

However, it is *total* energy that is conserved, and electromagnetic energy is only one form of energy.

In fact, there are two ways that the electromagnetic energy of a particular region can change. One way is for that energy to flow in or out of the region as electromagnetic energy. That possibility is well-described by the energy flux term on the right-hand side of the equation above. The second way is for the electric and magnetic fields in the region to do work on charges that are in the region (by virtue of the force they exert on the charges). This work done represents a transfer from electromagnetic energy to mechanical energy (or vice versa if work is negative). We will need to include a term for that in the equation above.

The rate of work done by a force acting on a particle is given by $\vec{F} \cdot \vec{v}$. If that particle has charge q, then the electric and magnetic fields will do work at a rate given by

$$P = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} = q\vec{E} \cdot \vec{v}$$

Note that the magnetic field doesn't do any work on the charge (since its force is perpendicular to $\vec{v}$), but the electric field can do work on moving charges.

If we consider a small volume dV of charge with density ρ moving with velocity $\vec{v}$, then the rate of work done on this charge is given by

$$P = (\rho\, dV)\vec{E} \cdot \vec{v} = \vec{E} \cdot (\rho\vec{v})\, dV = \vec{E} \cdot \vec{J}\, dV$$

Summing over all such charges within a given region yields

$$P = \int_\Omega \vec{E} \cdot \vec{J}\, dV$$

As a positive number, this power would represent a rate of production of mechanical energy. It follows that there would be a contribution of

$$-\int_\Omega \vec{E} \cdot \vec{J}\, dV$$

to the rate of change of electromagnetic energy. The full equation of continuity would then be given by

$$\frac{d}{dt}\int_\Omega u\, dV = -\oint_{\partial\Omega} \vec{S} \cdot d\vec{A} - \int_\Omega \vec{E} \cdot \vec{J}\, dV \tag{2.6}$$

which can also be expressed in differential form by shrinking the volume to a point and dividing out the volume:

$$\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S} - \vec{E} \cdot \vec{J} \tag{2.7}$$

This is the proper statement of conservation of energy in the context of electromagnetic energy.

2.3 Conservation of momentum

Momentum density:

$$\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \frac{1}{c^2}\vec{S} \tag{2.8}$$

Momentum current density (linear mapping / matrix):

$$\sigma = -\epsilon_0 \vec{E} \otimes \vec{E} - \frac{1}{\mu_0}\vec{B} \otimes \vec{B} + uI \tag{2.9}$$

$$\sigma_{ij} = -\epsilon_0 E_i E_j - \frac{1}{\mu_0} B_i B_j + u\delta_{ij}$$

where

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Rate of flow of momentum across $d\vec{A}$ is given by $\sigma d\vec{A}$ (evaluation of linear mapping; i-th component given by $\sum_j \sigma_{ij} dA_j$). Rate of flow across general (large) surface is $\int_\Sigma \sigma d\vec{A}$ (i-th component given by $\int_\Sigma \sum_j \sigma_{ij} dA_j$).

σ is a matrix because momentum is already a vector, and forming the current density of a quantity requires an additional vector component index (to combine with the surface vector d$\vec{A}$). With σ_{ij}, the i index (row index) specifies a momentum component and the j index (column index) specifies an area component. In other words, σ_{ij} is the rate per cross-sectional area of the i-th component of momentum flow across a surface perpendicular to the j-th coordinate axis.

Just as with energy, it is *total* momentum that is conserved, but the electromagnetic momentum (as defined above) is only one form of momentum. Moving particles also carry momentum, and momentum can be transfered from the electromagnetic fields to charged particles via the Lorentz Force (eq. 1.10):

$$\vec{F} = \int_\Omega (\rho\vec{E} + \vec{J} \times \vec{B})\, dV$$

Rate of momentum transfer to charged particles given by $\vec{F}$ itself (evident from Newton's Second Law, which can be written $\vec{F}_{\text{net}} = d\vec{p}/dt$). Therefore, rate of change in EM momentum is given by $-\vec{F}$.

Combining this with the fact that EM momentum can flow in or out of the region (as described by σ) yields the following equation of continuity:

$$\frac{d}{dt}\int_\Omega \vec{g}\, dV = -\oint_{\partial\Omega} \sigma d\vec{A} - \int_\Omega (\rho\vec{E} + \vec{J} \times \vec{B})\, dV \tag{2.10}$$

which can be expressed in differential form

$$\frac{\partial \vec{g}}{\partial t} = -\vec{\nabla} \cdot \sigma - (\rho E + \vec{J} \times \vec{B}) \tag{2.11}$$

where it is understood that $(\vec{\nabla} \cdot \sigma)_i = \sum_j \partial_j \sigma_{ij}$

2.4 Derivations

Derivations of eq. 2.7 and 2.11...

First, two identities:

$$
\begin{aligned}
\vec{\nabla} \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \\
\vec{\nabla} \cdot (-\vec{A} \otimes \vec{A} + \frac{1}{2} A^2 I) &= \vec{A} \times (\vec{\nabla} \times \vec{A}) - (\vec{\nabla} \cdot \vec{A}) \vec{A}
\end{aligned}
$$

Proved using "tensor notation" (see Appendix 3 for details).

This proves the first identity:

$$
\begin{aligned}
\vec{\nabla} \cdot (\vec{A} \times \vec{B}) &= \partial_i (\epsilon_{ijk} A_j B_k) \\
&= \epsilon_{ijk} \partial_i (A_j B_k) \\
&= \epsilon_{ijk} ((\partial_i A_j) B_k + A_j (\partial_i B_k)) \\
&= B_k (\epsilon_{ijk} \partial_i A_j) + A_j (\epsilon_{ijk} \partial_i B_k) \\
&= B_k (\epsilon_{kij} \partial_i A_j) - A_j (\epsilon_{jik} \partial_i B_k) \\
&= \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})
\end{aligned}
$$

This (clarifies and) proves the second identity:

$$
\begin{aligned}
(\vec{\nabla} \cdot (-\vec{A} \otimes \vec{A} + \frac{1}{2} A^2 I))_i &= \partial_j (-\vec{A} \otimes \vec{A} + \frac{1}{2} A^2 I)_{ij} \\
&= \partial_j (-A_i A_j + \frac{1}{2} A^2 \delta_{ij}) \\
&= \partial_j (-A_i A_j) + \partial_i (\frac{1}{2} A^2) \\
&= -\partial_j (A_i A_j) + \partial_i (\frac{1}{2} A_j A_j) \\
&= -(\partial_j A_i) A_j - A_i (\partial_j A_j) + \frac{1}{2} 2 A_j (\partial_i A_j) \\
&= A_j (\partial_i A_j - \partial_j A_i) - (\partial_j A_j) A_i \\
&= A_j (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}) \partial_{i'} A_{j'} - (\partial_j A_j) A_i \\
&= A_j (\epsilon_{kij} \epsilon_{ki'j'}) \partial_{i'} A_{j'} - (\partial_j A_j) A_i \\
&= \epsilon_{kij} A_j (\epsilon_{ki'j'} \partial_{i'} A_{j'}) - (\partial_j A_j) A_i \\
&= \epsilon_{ijk} A_j (\vec{\nabla} \times \vec{A})_k - (\vec{\nabla} \cdot \vec{A}) A_i \\
&= (\vec{A} \times (\vec{\nabla} \times \vec{A}))_i - (\vec{\nabla} \cdot \vec{A}) A_i \\
&= (\vec{A} \times (\vec{\nabla} \times \vec{A}) - (\vec{\nabla} \cdot \vec{A}) \vec{A})_i
\end{aligned}
$$

Proof of eq. 2.7:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{S} &= \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \\
&= \frac{1}{\mu_0}(\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})) \\
&= \frac{1}{\mu_0}\vec{B}\cdot(-\frac{\partial\vec{B}}{\partial t}) - \frac{1}{\mu_0}\vec{E}\cdot\mu_0(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}) \\
&= -(\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} + \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}) - \vec{E}\cdot\vec{J} \\
&= -\frac{\partial}{\partial t}(\frac{1}{2\mu_0}B^2 + \frac{1}{2}\epsilon_0 E^2) - \vec{E}\cdot\vec{J} \\
&= -\frac{\partial u}{\partial t} - \vec{E}\cdot\vec{J}
\end{aligned}$$

Simple rearrangement yields eq. 2.7. Note that Maxwell's equations (eq. 1.6 and 1.8 in particular) are substituted in between the second and third lines of the derivation above.

Proof of eq. 2.11:

$$\begin{aligned}
\vec{\nabla}\cdot\sigma &= \vec{\nabla}\cdot(-\epsilon_0\vec{E}\otimes\vec{E} - \frac{1}{\mu_0}\vec{B}\otimes\vec{B} + uI) \\
&= \epsilon_0\vec{\nabla}\cdot(-\vec{E}\otimes\vec{E} + \frac{1}{2}E^2 I) + \frac{1}{\mu_0}\vec{\nabla}\cdot(-\vec{B}\otimes\vec{B} + \frac{1}{2}B^2 I) \\
&= \epsilon_0(\vec{E}\times(\vec{\nabla}\times\vec{E}) - (\vec{\nabla}\cdot\vec{E})\vec{E}) + \frac{1}{\mu_0}(\vec{B}\times(\vec{\nabla}\times\vec{B}) - (\vec{\nabla}\cdot\vec{B})\vec{B}) \\
&= \epsilon_0\vec{E}\times(-\frac{\partial\vec{B}}{\partial t}) - \epsilon_0(\frac{\rho}{\epsilon_0})\vec{E} + \frac{1}{\mu_0}\vec{B}\times\mu_0(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}) - \frac{1}{\mu_0}(0)\vec{B} \\
&= -\epsilon_0\vec{E}\times\frac{\partial\vec{B}}{\partial t} + \epsilon_0\vec{B}\times\frac{\partial\vec{E}}{\partial t} - \rho\vec{E} + \vec{B}\times\vec{J} \\
&= -\epsilon_0(\vec{E}\times\frac{\partial\vec{B}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\times\vec{B}) - \rho\vec{E} - \vec{J}\times\vec{B} \\
&= -\epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \rho\vec{E} - \vec{J}\times\vec{B} \\
&= -\frac{\partial\vec{g}}{\partial t} - \rho\vec{E} - \vec{J}\times\vec{B}
\end{aligned}$$

Simple rearrangement yields eq. 2.11. Note that all four of Maxwell's equations are substituted in between the third and fourth lines of the derivation above.

At this point, we should consider what we have just accomplished. Let's focus on energy for the moment.

So here is the issue. If u and $\vec{S}$, as defined by eq. 2.4 and eq. 2.5, truly represent electromagnetic energy density and current density, shouldn't eq. 2.7 simply be a matter of conservation of energy? What do we accomplish by deriving the equation of continuity

from Maxwell's equations, when we know it must be true just from the point of view of conservation of energy?

But that is the point. How do we know that u and $\vec{S}$, as defined by eq. 2.4 and eq. 2.5, actually represent energy density and energy current density for the electromagnetic fields? I claim that they do. But why take my word for it?

The truth is, there is no direct way to prove eq. 2.4 and eq. 2.5, as there is no general definition of energy that doesn't, at least on the surface, appear to be circular. However, by deriving the equation of continuity directly from Maxwell's equations, we are actually justifying our definition of energy density and energy current density using equations 2.4 and 2.5.

Here is a way to look at this. First of all, we know that u and $\vec{S}$, as we have defined them, have the correct *units* for energy density and energy current density, so we can imagine that they represent *some* form of energy. Perhaps it is not appropriate to label them as representing *electromagnetic energy*, but surely they represent something.

Likewise, we would have to imagine that there is some "true" definition of electromagnetic energy density and current density given by u_{true} and $\vec{S}_{\text{true}}$. At this point, we don't know what the formulas look like. Perhaps they are identical to u and $\vec{S}$. Perhaps they are not.

But anyway, if u_{true} and $\vec{S}_{\text{true}}$ truly represent electromagnetic energy density and current density, then the following equation of continuity

$$\frac{\partial u_{\text{true}}}{\partial t} = -\vec{\nabla} \cdot \vec{S}_{\text{true}} - \vec{E} \cdot \vec{J}$$

must be valid for the simple reason that (total) energy is conserved and electromagnetic energy can only change as a result of electric and magnetic fields doing work on charges via the Lorentz force given by eq. 1.10.

Likewise, we have established the following equation of continuity

$$\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S} - \vec{E} \cdot \vec{J}$$

for u and $\vec{S}$ as defined by eq. 2.4 and 2.5. This was proved directly from Maxwell's equations, and so there is no controversy there.

We can now define $u_{\text{diff}} = u_{\text{true}} - u$ and $\vec{S}_{\text{diff}} = \vec{S}_{\text{true}} - \vec{S}$. If we then subtract the two equations above, we are left with

$$\frac{\partial u_{\text{diff}}}{\partial t} = -\vec{\nabla} \cdot \vec{S}_{\text{diff}}$$

It should be noted that the $\vec{E} \cdot \vec{J}$ terms cancel out, since they are in both equations.

To summarize, we are considering three forms of energy which we will call "proposed electromagnetic energy" (density and current density given by u and $\vec{S}$), "true electromagnetic energy" (density and current density given by u_{true} and $\vec{S}_{\text{true}}$), and "differential energy" (density and current density given by u_{diff} and $\vec{S}_{\text{diff}}$). Differential energy represents the difference between true electromagnetic energy and our proposed value of electromagnetic energy.

It is clear from the equation of continuity for differential energy that differential energy is, *by itself*, a universally conserved quantity. There is no transfer mechanism for converting differential energy into any other form of energy. This raises the question of how we can actually observe this form of energy? How do we measure differential energy? The answer is that we cannot. This form of energy exists in its own universe. It can move around ($\vec{S}_{\text{diff}}$ describes that motion), but cannot affect anything.

So this now gets to the real question — how do we choose to model the universe? The goal of constructing these scientific models is to come up with some sort of quantitative framework for being able to analyze the universe and make predictions about future behavior of various systems. This differential energy will have no impact on this analysis.

In light of this fact, we may as well get rid of this differential energy by setting u_{diff} and $\vec{S}_{\text{diff}}$ to zero. In doing this, we find that $u = u_{\text{true}}$ and $\vec{S} = \vec{S}_{\text{true}}$, and thus our proposed formulas for electromagnetic energy density and energy current density, in fact, truly represent electromagnetic energy. Note that all of this analysis follows from the derivation of the appropriate equation of continuity (eq. 2.7) from fundamental principles (which, in this case, is Maxwell's equations and the Lorentz force law).

One way to succinctly summarize this discussion is to say the following: if a quantity "acts" like electromagnetic energy (in the sense of obeying an appropriate equation of continuity), then it must *be* electromagnetic energy. In any case, we cannot go wrong by assuming this.

This same argument can be applied to electromagnetic momentum. The fact that $\vec{g}$ and σ, as defined by eq. 2.8 and eq. 2.9, satisfy the continuity equation (eq. 2.11), as we proved above using Maxwell's equations, provides adequate justification for asserting that $\vec{g}$ and σ, as we have already defined it, truly represents electromagnetic momentum density and momentum current density.

I want to finish this section by considering the following definition of energy that I have seen in the literature: that energy is defined as "the ability to do work". In the past, I have always regarded this definition of energy to be circular since the same scientists will then invariably define work as a transfer of energy through a mechanical force.

However, if one carefully considers the argument we made above, we have essentially defined electromagnetic energy in terms of the electric and magnetic field's ability to do work on charges. Certainly *kinetic* energy is quite measureable, since motion is so easily defined and observed. It would make sense that any quantity that has units of energy, and can be transferred to kinetic energy can be considered an actual form of energy which should be added to what we would consider to be "total energy" (a conserved quantity).

Perhaps this definition of energy (ability to do work) isn't as circular as I had once thought.

3 The Potentials

3.1 Scalar and vector potentials

It is a simple matter of calculation that the curl of any gradient is zero. Likewise, the divergence of any curl is also zero.

$$
\begin{aligned}
\vec{\nabla} \times (\vec{\nabla}\phi) &= 0 \\
\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= 0
\end{aligned}
$$

Both identities are based on the antisymmetric properties of cross products, along with the commutativity of derivatives.

Perhaps less obvious is the converse, that any vector field that has zero divergence can be written as a the curl of another vector field, and that any vector field that has zero curl can be written as (minus) the gradient of a scalar field.

On the basis of these two results, along with Maxwell's equations, we can express the electric and magnetic fields, both vector fields (6 components total), in terms of a vector field and a scalar field (4 components total).

First consider Gauss' Law for the magnetic field (eq. 1.7):

$$\vec{\nabla} \cdot \vec{B} = 0$$

On the basis of this equation, we may write $\vec{B}$ as the curl of another vector field, $\vec{A}$, which is known as the *vector potential*:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Now consider Faraday's Law (eq. 1.6):

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

In the absence of time-varying magnetic fields, $\vec{E}$ has zero curl (it is a conservative field) and can be written as minus the gradient of a scalar field. No doubt, students who have taken an introductory physics course in electromagnetism can recall the *electric potential* (often written V) which is defined by

$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s}$$

Since the electric field is a conservative field, the line integral above depends only on the endpoints and not the integration path, and so this definition of V (up to an arbitrary additive constant) is justified. It should be noted that this defining equation for V is equivalent to the assertion that $\vec{E} = -\vec{\nabla}V$.

It turns out that the electric force (being proportional to $\vec{E}$) is also conservative (in the physics sense), and thus a potential energy function can be defined by

$$U_b - U_a = -W^{(\vec{E})}_{a\to b} = -\int_a^b \vec{F}\cdot d\vec{s}$$

Since $\vec{F} = q\vec{E}$, we have $U = qV$, and so the electric potential is intimately related to the potential energy of charges moving around in an electric field.

In the presence of a time-varying magnetic field, the electric field is no longer curl-free. However, if we plug $\vec{B} = \vec{\nabla}\times\vec{A}$ into Faraday's Law, we find that

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

A little bit of rearranging yields

$$\vec{\nabla}\times(\vec{E} + \frac{\partial\vec{A}}{\partial t}) = 0$$

It follows that $E + \partial\vec{A}/\partial t$ *is* curl-free, even in the general case, and therefore can be written as (minus) the gradient of a scalar field, ϕ, which is known as the *scalar potential*:

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

Solving for $\vec{E}$ yields

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

In the absence of time-varying magnetic fields ($\partial\vec{A}/\partial t = 0$), the electric field will again be conservative, and will be given by $-\vec{\nabla}\phi$. The scalar potential will be none other than the electric potential.

It should be noted that the relationship between the scalar potential and potential energy is far less clear in the general case where time-varying magnetic fields exist. Furthermore, I am not aware of any meaningful relationship between potential energy and the vector potential. Keep in mind that magnetic fields never do work on charges.

Putting these results together, we find that $\vec{E}$ and $\vec{B}$ can be expressed in terms of a scalar potential ϕ and vector potential $\vec{A}$ as follows:

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{3.1}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \tag{3.2}$$

Although the connection between the potentials (both ϕ and $\vec{A}$) and potential energy of charges is lost in the general case, the method of calculating ϕ and $\vec{A}$ from the charge and current distributions will be eerily familiar to introductory physics students who have seen this done for ϕ in the electrostatic case. We will address this in the next few sections.

3.2 Gauge transformations

Note: electric potential was defined up to a constant: If V_1 and V_2 are potential functions that differ by a constant, then $\vec{E} = -\vec{\nabla}V_1 = -\vec{\nabla}V_2$ (same field). To disambiguate the potential, we established a reference value of V at some reference point.

More flexibility with both a scalar and vector potential.

Gauge transformations:

$$\vec{A}' = \vec{A} + \vec{\nabla}\psi \qquad \phi' = \phi - \frac{\partial\psi}{\partial t} \tag{3.3}$$

where ψ is an arbitrary scalar field.

$$\begin{aligned}
\vec{\nabla} \times \vec{A}' &= \vec{\nabla} \times (\vec{A} + \vec{\nabla}\psi) \\
&= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}\psi) \\
&= \vec{\nabla} \times \vec{A} \\
&= \vec{B}
\end{aligned}$$

$$\begin{aligned}
-\vec{\nabla}\phi' - \frac{\partial\vec{A}'}{\partial t} &= -\vec{\nabla}(\phi - \frac{\partial\psi}{\partial t}) - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\psi) \\
&= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}(\frac{\partial\psi}{\partial t}) - \frac{\partial}{\partial t}(\vec{\nabla}\psi) \\
&= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \\
&= \vec{E}
\end{aligned}$$

Evidently, potential sets $(\vec{A}', \phi')$ and $(\vec{A}, \phi)$ both generate the same electric and magnetic fields.

To help remove ambiguity, we often impose a *gauge condition*. There are a number of them, but one of the most popular is the Lorentz gauge:[1]

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \tag{3.4}$$

To impose this gauge, we first note that

$$\begin{aligned}
\vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t} &= \vec{\nabla} \cdot (\vec{A} + \vec{\nabla}\psi) + \frac{1}{c^2}\frac{\partial}{\partial t}(\phi - \frac{\partial\psi}{\partial t}) \\
&= \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}
\end{aligned}$$

[1] Wikipedia claims that this gauge condition should actually be named after Ludvig Lorenz (no "t" in the last name) rather than Hendrik Lorentz (who is famous for his role in the development of relativity), although the situation is quite complicated. Anyway, all three of my go-to sources (Feynmann, Jackson, and Lorrain/Corson/Lorrain (my upper-division EM course textbook)) all attribute this gauge condition to Lorentz (with the "t"). So that's what I will do until I am convinced to do otherwise.

If we choose some random $(\vec{A}, \phi)$ to satisfy eqs. 3.1 and 3.2, it is likely that the Lorentz gauge condition will *not* be satisfied:

$$\alpha(\vec{r}, t) = \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

will not be zero. However, if we choose ψ so that

$$\nabla^2 \psi + \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -\alpha(\vec{r}, t)$$

then, according to the derivation above, $(\vec{A}', \phi')$ will (1) generate the same electric and magnetic fields (because these potentials are a gauge transformation of the original potentials), and (2) satisfy the Lorentz gauge condition (eq. 3.4).

Note that the following field ψ solves the differential equation (see eqs. D.6 and D.7 in Appendix 2):

$$\psi(\vec{r}, t) = +\frac{1}{4\pi} \int \frac{\alpha(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c)}{|\vec{r} - \vec{r}\,'|}\, dV'$$

It follows that the Lorentz gauge condition can always be imposed.

We will find this convenient as we continue in our quest to solve for $\vec{A}$ and ϕ (and therefore $\vec{E}$ and $\vec{B}$) for a given charge and current distribution.

3.3 Potentials due to charge/current distributions

Electric and magnetic fields calculated from potentials using eqs. 3.1 and 3.2 will automatically satisfy the second and third Maxwell's equations (eqs. 1.6 and 1.7). To ensure that the first and fourth equations (eqs. 1.5 and 1.8) are satisfied, we plug eqs. 3.1 and 3.2 into those equations and turn the crank.

From Gauss' Law (eq. 1.5):

$$\begin{aligned}
\frac{\rho}{\epsilon_0} &= \vec{\nabla} \cdot \vec{E} \\
&= \vec{\nabla} \cdot (-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}) \\
&= -\nabla^2 \phi - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) \\
&= -\nabla^2 \phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t})
\end{aligned}$$

From Maxwell-Ampere's Law (eq. 1.8):

$$
\begin{aligned}
\mu_0 \vec{J} &= \vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\
&= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \frac{1}{c^2} \frac{\partial}{\partial t} (-\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}) \\
&= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} - \frac{1}{c^2} \vec{\nabla} (\frac{\partial \phi}{\partial t}) + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \\
&= -\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} + \vec{\nabla} (\vec{\nabla} \cdot \vec{A} - \frac{1}{c^2} \frac{\partial \phi}{\partial t})
\end{aligned}
$$

Some simple manipulations yield:

$$
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}) = -\frac{\rho}{\epsilon_0} \tag{3.5}
$$

$$
\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} (\vec{\nabla} \cdot \vec{A} - \frac{1}{c^2} \frac{\partial \phi}{\partial t}) = -\mu_0 \vec{J} \tag{3.6}
$$

It turns out that finding a solution to Maxwell's equations is equivalent to solving eqs. 3.5 and 3.6 for $\vec{A}(\vec{r}, t)$ and $\phi(\vec{r}, t)$ and then plugging this solution into eqs. 3.1 and 3.2 to find $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$.

Eqs. 3.5 and 3.6 are very complicated differential equations, with $\vec{A}$ and ϕ appearing in both equations. However, if we impose the Lorentz gauge condition on $(\vec{A}, \phi)$, then Eqs. 3.5 and 3.6 reduce to

$$
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{3.7}
$$

$$
\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{3.8}
$$

These equations are much simpler to solve: they are decoupled. In fact, eq. 3.8 can be separated into three separate component equations:

$$
\nabla^2 A_i - \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2} = -\mu_0 J_i \qquad (i = x,\ y,\ \text{or } z)
$$

Furthermore, each of these differential equations are wave equations with a source function. The following is a solution (see eqs. D.6 and D.7 in Appendix 2):

$$
\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c)}{|\vec{r} - \vec{r}\,'|} \, dV' \tag{3.9}
$$

$$
\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c)}{|\vec{r} - \vec{r}\,'|} \, dV' \tag{3.10}
$$

Note the implication here: both the scalar/vector potential at a given field point and time $(\vec{r}, t)$ is determined from the charge density / current density at various points in space

$\vec{r}'$, but not at time t, but at an earlier time $t - |\vec{r} - \vec{r}'|/c$. Presumably the source charge / current sends a signal at the earlier "retarded time" which travels at speed c and reaches the field point at time t.

It should be noted that there is a decoupling which occurs with the potentials, in that ϕ depends only on ρ, and $\vec{A}$ depends only on $\vec{J}$. This is in contrast with the electric field, which may depend on both ρ and $\vec{J}$ — the latter can occur if the current produces a time-varying magnetic field which acts as a source of electric field (Faraday's Law).

3.4 Verifying the potentials

Let us first begin by verifying the Lorentz gauge condition for the potentials $\vec{A}$ and ϕ given by eqs. 3.9 and 3.10.

At first glance, it appears that this will not be difficult, as $\vec{A}$ depends on $\vec{J}$ in pretty much the same way that ϕ depends on ρ, and the Lorentz condition (eq. 3.4)

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

very much looks like the equation of continuity (eq. 2.3)

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

which follows from conservation of charge. In fact, it looks like the constants are behaving since

$$\frac{\mu_0}{4\pi} = \frac{1}{c^2}\frac{1}{4\pi\epsilon_0}$$

Let's first look at the time derivative of ϕ:

$$\frac{\partial \phi}{\partial t} = \frac{1}{4\pi\epsilon_0}\int \frac{\partial}{\partial t}\left(\frac{\rho(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|}\right) dV'$$

This will not be problematic. The only part of the integrand that depends on t is in the time argument of ρ. Furthermore, the chain rule is quite simple:

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(\vec{r}', t - |\vec{r} - \vec{r}'|/c) &= \frac{\partial \rho}{\partial t'}\Big|_{t'=t-|\vec{r}-\vec{r}'|/c}\frac{\partial}{\partial t}(t - |\vec{r} - \vec{r}'|/c) \\
&= \frac{\partial \rho}{\partial t'}\Big|_{t'=t-|\vec{r}-\vec{r}'|/c}(1) \\
&= \frac{\partial \rho}{\partial t'}\Big|_{t'=t-|\vec{r}-\vec{r}'|/c}
\end{aligned}$$

It therefore appears that $\partial\phi/\partial t$ depends on $\partial\rho/\partial t'$ in exactly the same way that ϕ depends on ρ.

However, the divergence of $\vec{A}$ looks like it may get us into trouble:

$$\vec{\nabla} \cdot \vec{A} = \frac{\mu_0}{4\pi}\int \vec{\nabla} \cdot \left(\frac{\vec{J}(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|}\right) dV'$$

First of all, it should be noted that $\vec{r}'$ represents the source point and is being integrated over — this variable is independent of $\vec{r}$, which represents the field point. It seems that the $\vec{r}$ dependence of the integrand is very complicated, as the time argument of $\vec{J}$ depends on $\vec{r}$, as well as the $1/|\vec{r}-\vec{r}'|$ factor. In fact, the only place where there is no $\vec{r}$ dependence is in the spatial argument of $\vec{J}$, which is precisely where we want $\vec{r}$ dependence. It looks like we are in for a rough ride.

It is at this point that we make the following substitution:

$$\vec{u} = \vec{r}' - \vec{r} \qquad dV_{\vec{u}} = dV_{\vec{r}'}$$

and suddenly everything falls into place.

First note that this substitution is simply a translational shift, so the volume elements are indeed unaffected by the substitution. Furthermore, $\vec{r}' = \vec{r} + \vec{u}$ and $|\vec{r} - \vec{r}'| = |\vec{u}|$ (the reversal of subtraction is immaterial), and so we get

$$\begin{aligned}
\phi(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}+\vec{u},t-|\vec{u}|/c)}{|\vec{u}|}\,dV_{\vec{u}} \\
\vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}+\vec{u},t-|\vec{u}|/c)}{|\vec{u}|}\,dV_{\vec{u}}
\end{aligned}$$

Both sets of derivatives will behave:

$$\begin{aligned}
\frac{\partial\phi}{\partial t}(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\frac{\partial\rho}{\partial t'}(\vec{r}+\vec{u},t-|\vec{u}|/c)}{|\vec{u}|}\,dV_{\vec{u}} \\
\vec{\nabla}\cdot\vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{J}(\vec{r}+\vec{u},t-|\vec{u}|/c)}{|\vec{u}|}\,dV_{\vec{u}}
\end{aligned}$$

And now we combine them:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} &= \frac{\mu_0}{4\pi}\int \frac{(\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t'})(\vec{r}+\vec{u},t-|\vec{u}|/c)}{|\vec{u}|}\,dV_{\vec{u}} \\
&= \frac{\mu_0}{4\pi}\int \frac{0}{|\vec{u}|}\,dV_{\vec{u}} \\
&= 0
\end{aligned}$$

At this point, we can verify that ϕ and $\vec{A}$, as defined by eqs. 3.9 and 3.10, will yield a solutioh to Maxwell's equations after substituting ϕ and $\vec{A}$ into eqs. 3.1 and 3.2.

After all, we know from Appendix 2 (see eqs. D.6 and D.7) that ϕ and $\vec{A}$ are solutions to eqs. 3.7 and 3.8. Likewise, since we have now confirmed the Lorentz gauge condition, we can also conclude that ϕ and $\vec{A}$ are solutions to eqs. 3.5 and 3.6. And those equations can be traced back to Maxwell's equations after substituting ϕ and $\vec{A}$ into eqs. 3.1 and 3.2.

Example of ϕ and $\vec{A}$ which satisfy eqs. 3.7 and 3.8, but not the Lorentz gauge condition, and therefore will *not* lead to a correct solution to Maxwell's equations. Just add a plane wave to ϕ and not $\vec{A}$. This will add a plane wave to $\vec{E}$ with no corresponding wave to $\vec{B}$.

4 Time Independent Fields

4.1 Electrostatics and magnetostatics

What if ρ and $\vec{J}$ do not depend on time?

5 Potentials Due to Point Charges

5.1 Point charges

A rigid charge distribution is characterized by a charge density function $\bar{\rho}(\vec{r})$ which gives the charge density of the charge distribution relative to a reference point at $\vec{r} = 0$. The reference point itself moves around relative to a fixed coordinate system according to the position function $\vec{R}(t)$. It follows that the charge distribution is given by

$$\rho(\vec{r}, t) = \bar{\rho}(\vec{r} - \vec{R}(t)) \tag{5.1}$$

The velocity and acceleration of the charge distribution can be calculated in the usual way:

$$\vec{v}(t) = \frac{d\vec{R}}{dt} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{R}}{dt^2} \tag{5.2}$$

Since the charge distribution moves rigidly with velocity $\vec{v}$, the current density is given by

$$\vec{J}(\vec{r}, t) = \vec{v}(t)\rho(\vec{r}, t) = \vec{v}(t)\bar{\rho}(\vec{r} - \vec{R}(t)) \tag{5.3}$$

We can check the equation of continuity:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \vec{\nabla}\bar{\rho}(\vec{r} - \vec{R}(t)) \cdot \frac{\partial}{\partial t}(\vec{r} - \vec{R}(t)) \\ &= \vec{\nabla}\bar{\rho}(\vec{r} - \vec{R}(t)) \cdot (-\vec{v}(t)) \end{aligned}$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{J} &= \vec{\nabla} \cdot (\vec{v}(t)\bar{\rho}(\vec{r} - \vec{R}(t)) \\ &= \vec{v}(t) \cdot \vec{\nabla}\bar{\rho}(\vec{r} - \vec{R}(t)) \end{aligned}$$

Adding these two equations yields

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

A point charge q is a rigid charge distribution with

$$\bar{\rho}(\vec{r}) = q\delta(\vec{r})$$

Indeed, the net charge is given by

$$\int \bar{\rho}(\vec{r})\, dV = \int q\delta(\vec{r})\, dV = q$$

which checks out.

If this charge moves through space according to $\vec{R}(t)$ (with corresponding velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$ given by eq. 5.2), then the charge density and current density functions are given by

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{R}(t)) \tag{5.4}$$

$$\vec{J}(\vec{r},t) = q\vec{v}(t)\delta(\vec{r}-\vec{R}(t)) \tag{5.5}$$

Our goal in the next few sections is to calculate first the scalar and vector potentials, and then the electric and magnetic fields due to a single charge moving arbitrarily.

5.2 Potentials due to point charge

The scalar and vector potential due to a point charge q undergoing arbitrary motion given by $\vec{R}(t)$ (with velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$) is obtained by substituting eqs. 5.4 and 5.5 into eqs. 3.9 and 3.10, respectively:

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q\delta(\vec{r}\,'-\vec{R}(t-|\vec{r}-\vec{r}\,'|/c))}{|\vec{r}-\vec{r}\,'|}\,dV' \tag{5.6}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{q\vec{v}(t-|\vec{r}-\vec{r}\,'|/c)\delta(\vec{r}\,'-\vec{R}(t-|\vec{r}-\vec{r}\,'|/c))}{|\vec{r}-\vec{r}\,'|}\,dV' \tag{5.7}$$

For the remainder of this section, we will consider a given fixed value of the field point $\vec{r}$ and field time (or “present time”) t.

It should be noted that both of these integrals are of the form

$$\int f(\vec{r}\,')\delta(\vec{g}(\vec{r}\,'))\,dV'$$

where

$$\vec{g}(\vec{r}\,') = \vec{r}\,' - \vec{R}(t-|\vec{r}-\vec{r}\,'|/c) \tag{5.8}$$

According to eq. C.2 (see Appendex 1), the value of this integral is given by

$$f(\vec{r}_0)/|\det\Gamma|$$

where $\vec{g}(\vec{r}_0)=0$ (unique solution) and $\Gamma_{ij}=\partial'_j g_i$ evaluated at $\vec{r}\,'=\vec{r}_0$. Thus, our next goal is to determine $\vec{r}_0$ and the matrix Γ.

If $\vec{g}(\vec{r}_0)=0$, then it must be the case that $\vec{r}_0=\vec{R}(t')$, where

$$t' = t-|\vec{r}-\vec{r}_0|/c = t-|\vec{r}-\vec{R}(t')|/c$$

It turns out that (for fixed $\vec{r}$ and t) there is exactly one time t' satisfying this equation above. We will denote this time as T, and refer to it as the “retarded time” associated with the particle. It is given implicitly as a function of $\vec{r}$ and t by

$$T = t-|\vec{r}-\vec{R}(T)|/c \tag{5.9}$$

In that case $\vec{r}_0 = \vec{R}(T)$, and is the only solution to $\vec{g}(\vec{r}_0) = 0$. As a check:

$$\begin{aligned}
\vec{g}(\vec{r}_0) &= \vec{g}(\vec{R}(T)) \\
&= \vec{R}(T) - \vec{R}(t - |\vec{v} - \vec{R}(T)|/c) \\
&= \vec{R}(T) - \vec{R}(T) \\
&= 0
\end{aligned}$$

Physically, the retarded time can be determined by imagining a spherical wave pulse eminating from the field point $\vec{r}$ at time t and then travelling *backwards in time* and away from $\vec{r}$ with speed c. If we follow the charge trajectory backwards in time from t, it will move with a speed that is less than c (massless charged particles have never been observed), and will therefore be overtaken by the spherical pulse at some time T. Once the pulse catches the charge and passes it, the pulse will continue to recede away faster than the charge can move, and so the charge will never encounter the pulse again. This uniquely determines T.

If we run this entire show forwards in time, a "field signal" will be emitted from the charge at time T from location $\vec{R}(T)$ and directed towards $\vec{r}$. This signal will travel at speed c for a time $|\vec{r} - \vec{R}(T)|/c$ (covering a distance $|\vec{r} - \vec{R}(T)|$), whereupon the signal reaches the field point $\vec{r}$ at time t, contributing to the scalar and vector potentials at that point.

At this point, we wish to evaluate Γ_{ij}, which is equal to $\partial'_j g_i$ evaluated at $\vec{r}_0$. Just to be clear, ∂'_j presents the derivative with respect to r'_j, with all other components of $\vec{r}\,'$, as well as all components of $\vec{r}$ and t regarded as constants.

We begin with the following:

$$\begin{aligned}
|\vec{r} - \vec{r}\,'|\partial'_j(|\vec{r} - \vec{r}\,'|) &= \partial'_j(\frac{1}{2}|\vec{r} - \vec{r}\,'|^2) \\
&= \partial'_j(\frac{1}{2}(\vec{r} - \vec{r}\,') \cdot (\vec{r} - \vec{r}\,')) \\
&= (\vec{r} - \vec{r}\,') \cdot \partial'_j(\vec{r} - \vec{r}\,') \\
&= (\vec{r} - \vec{r}\,') \cdot (-\hat{e}_j) \\
&= -(r_j - r'_j)
\end{aligned}$$

from which we conclude that

$$\partial'_j(|\vec{r} - \vec{r}\,'|) = -\frac{r_j - r'_j}{|\vec{r} - \vec{r}\,'|}$$

It follows that

$$\begin{aligned}
\partial'_j g_i(\vec{r}\,') &= \partial'_j(r'_i - R_i(t - |\vec{r} - \vec{r}\,'|/c)) \\
&= \delta_{ij} - v_i(t - |\vec{r} - \vec{r}\,'|/c)\partial'_j(t - |\vec{r} - \vec{r}\,'|/c) \\
&= \delta_{ij} - v_i(t - |\vec{r} - \vec{r}\,'|/c)(-1/c)(-(r_j - r'_j)/|\vec{r} - \vec{r}\,'|) \\
&= \delta_{ij} - \frac{v_i(t - |\vec{r} - \vec{r}\,'|/c)}{c}\frac{r_j - r'_j}{|\vec{r} - \vec{r}\,'|}
\end{aligned}$$

If we define

$$\hat{n} = \frac{\vec{r} - \vec{R}(T)}{|\vec{r} - \vec{R}(T)|}$$

and plug $\vec{r}\,' = \vec{r}_0 = \vec{R}(T)$ into the above equation for $\partial'_j g_i$, then we get

$$\begin{aligned} \Gamma_{ij} &= \delta_{ij} - \frac{v_i(t - |\vec{r} - \vec{R}(T)|/c)}{c} \frac{r_j - R_j(T)}{|\vec{r} - \vec{R}(T)|} \\ &= \delta_{ij} - \frac{v_i(T)}{c} \hat{n}_j \end{aligned}$$

and so

$$\Gamma = I - \frac{\vec{v}(T)}{c} \otimes \hat{n}$$

The easiest way to evaluate the determinant of Γ is to first orient the coordinate system so that $\hat{n}$ lines up with the $+x$ direction: thus $\hat{n} = \hat{x}$. In that case,

$$\Gamma = \begin{pmatrix} 1 - v_x(T)/c & 0 & 0 \\ -v_y(T)/c & 1 & 0 \\ -v_z(T)/c & 0 & 1 \end{pmatrix}$$

The matrix is lower-triangular, and so the determinant is simply the product of the diagonal elements. Thus

$$\det(\Gamma) = (1 - v_x(T)/c)(1)(1) = 1 - v_x(T)/c = 1 - (\vec{v}(T)/c) \cdot \hat{n}$$

The last formula must be correct in all coordinate systems (whether $\hat{n}$ is lined up with $\hat{x}$ or not), since the expression is inherently coordinate-system invariant.

It follows that for any (scalar or vector) function $f(\vec{r}\,')$,

$$\int f(\vec{r}\,')\delta(\vec{g}(\vec{r}\,'))\, dV' = \beta f(\vec{R}(T))$$

where β is defined by

$$\beta = \frac{1}{\det \Gamma} = \frac{1}{1 - (\vec{v}(T)/c) \cdot \hat{n}} \tag{5.10}$$

Plugging this result into eqs. 5.6 and 5.7 yields:

$$\begin{aligned} \phi(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{q\delta(\vec{g}(\vec{r}\,'))}{|\vec{r} - \vec{r}\,'|}\, dV \\ &= \frac{1}{4\pi\epsilon_0} \frac{\beta q}{|\vec{r} - \vec{R}(T)|} \end{aligned}$$

$$\begin{aligned} \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{q\vec{v}(t - |\vec{r} - \vec{r}\,'|/c)\delta(\vec{g}(\vec{r}\,'))}{|\vec{r} - \vec{r}\,'|}\, dV' \\ &= \frac{\mu_0}{4\pi} \frac{\beta q \vec{v}(t - |\vec{r} - \vec{R}(T)/c)}{|\vec{r} - \vec{R}(T)|} \\ &= \frac{\mu_0}{4\pi} \frac{\beta q \vec{v}(T)}{|\vec{r} - \vec{R}(T)|} \end{aligned}$$

The final result:

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\beta q}{|\vec{r}-\vec{R}(T)|} \tag{5.11}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\beta q\vec{v}(T)}{|\vec{r}-\vec{R}(T)|} \tag{5.12}$$

where the retarded time T is determined implicitly (as a function of $\vec{r}$ and t) by eq. 5.9 and β is defined by eq. 5.10.

This result shouldn't be too surprising, except for the β factor, which you may not have been expecting. This factor arises physically because any finite-sized charge distribution has points that will be located at slightly different distances away from the field point, and should therefore be sampled at slightly different "retarded times". Since the charge is moving, the reference point will be located at slightly different locations as the contribution to the potentials from each part of the charge is calculated, and so the volume over which ρ is integrated will end up being distorted. This results in a volume factor, β, which does *not* go away in the limit as the size of the charge goes to zero.

The Feynmann Lecture Notes (volume II, section 21-5, pages 9–11) has some additional details, including pictures showing the case where the charge is moving directly towards the field point and a detailed calculation of the volume factor in that situation.

6 Fields Due to Point Charges

6.1 Fields due to point charge (setup)

At this point, with the scalar and vector potentials given by eqs. 5.11 and 5.12, it should be a "simple" matter of plugging those potentials into

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

and calculating the derivatives.

It turns out that the calculation is actually quite complicated, and needs to be done carefully to avoid errors. To set the stage, we first recall that the retarded time T is defined implicitly by

$$T = t - |\vec{r}-\vec{R}(T)|/c$$

To help facilitate the calculations, we will set $c = 1$ (choice of units), write $\vec{v}$ for $\vec{v}(T)$

and $\vec{a}$ for $\vec{a}(T)$, and make the following definitions:

$$\begin{aligned}
\vec{h} &= \vec{r} - \vec{R}(T) \\
h &= |\vec{h}| \\
\hat{n} &= \vec{h}/h \\
v_n &= \vec{v} \cdot \hat{n} \\
a_n &= \vec{a} \cdot \hat{n} \\
\beta &= 1/(1 - v_n) \\
\psi &= \beta/h
\end{aligned}$$

Note that β and $\hat{n}$ were defined in the last section and are defined equivalently here, although the definitions look much simpler in the new notation.

Everything that we defined above should be regarded as functions of $\vec{r}$ and t (field point and time).

The first order of business is to calculate derivatives of T, $\vec{h}$, and h. These functions are all tied together because of the implicit definition of T

$$T = t - h$$

where h, in turn, depends on T.

It is possible to untangle this:

$$\begin{aligned}
\partial_\mu \vec{h} &= \partial_\mu(\vec{r} - \vec{R}(T)) \\
&= \partial_\mu \vec{r} - \vec{v}\partial_\mu T
\end{aligned}$$

$$\begin{aligned}
h\partial_\mu h &= \partial_\mu((1/2)h^2) \\
&= \partial_\mu((1/2)\vec{h} \cdot \vec{h}) \\
&= \vec{h} \cdot \partial_\mu \vec{h}
\end{aligned}$$

$$\begin{aligned}
\partial_\mu h &= (\vec{h}/h) \cdot \partial_\mu \vec{h} \\
&= \hat{n} \cdot (\partial_\mu \vec{r} - \vec{v}\partial_\mu T) \\
&= \hat{n} \cdot \partial_\mu \vec{r} - v_n \partial_\mu T
\end{aligned}$$

$$\begin{aligned}
\partial_\mu T &= \partial_\mu(t - h) \\
&= \partial_\mu t - \partial_\mu h \\
&= \partial_\mu t - \hat{n} \cdot \partial_\mu \vec{r} + v_n \partial_\mu T
\end{aligned}$$

At this point, we can solve for $\partial_\mu T$:

$$\begin{aligned}
\partial_\mu T &= \partial_\mu t - \hat{n} \cdot \partial_\mu \vec{r} + v_n \partial_\mu T \\
\partial_\mu T - v_n \partial_\mu T &= \partial_\mu t - \hat{n} \cdot \partial_\mu \vec{r} \\
(1 - v_n)\partial_\mu T &= \partial_\mu t - \hat{n} \cdot \partial_\mu \vec{r} \\
\partial_\mu T &= \beta(\partial_\mu t - \hat{n} \cdot \partial_\mu \vec{r})
\end{aligned}$$

We can now separately calculate spatial and time derivatives of T, $\vec{h}$, and h.

$$\begin{aligned}
\partial_t T &= \beta(\partial_t t - \hat{n} \cdot \partial_t \vec{r}) \\
&= \beta(1 - \hat{n} \cdot \vec{0}) \\
&= \beta
\end{aligned}$$

$$\begin{aligned}
\partial_i T &= \beta(\partial_i t - \hat{n} \cdot \partial_i \vec{r}) \\
&= \beta(0 - \hat{n} \cdot \hat{e}_i) \\
&= -\beta n_i
\end{aligned}$$

$$\begin{aligned}
\partial_t \vec{h} &= \partial_t \vec{r} - \vec{v} \partial_t T \\
&= 0 - \vec{v}(\beta) \\
&= -\beta \vec{v}
\end{aligned}$$

$$\begin{aligned}
\partial_i \vec{h} &= \partial_i \vec{r} - \vec{v} \partial_i T \\
&= \hat{e}_i - \vec{v}(-\beta n_i) \\
&= \hat{e}_i + \beta n_i \vec{v}
\end{aligned}$$

$$\begin{aligned}
\partial_t h &= \partial_t (t - T) \\
&= 1 - \beta \\
&= -\beta v_n
\end{aligned}$$

$$\begin{aligned}
\partial_i h &= \partial_i (t - T) \\
&= 0 - (-\beta n_i) \\
&= \beta n_i
\end{aligned}$$

The following is a summary:

$$\partial_t T = \beta \qquad \vec{\nabla} T = -\beta \hat{n}$$

$$\partial_t \vec{h} = -\beta \vec{v} \qquad \partial_i \vec{h} = \hat{e}_i + \beta n_i \vec{v}$$

$$\partial_t h = 1 - \beta = -\beta v_n \qquad \vec{\nabla} h = \beta \hat{n}$$

Just a few more derivatives to go. Note that

$$\partial_\mu \vec{v} = \vec{a} \partial_\mu T$$

which leads to

$$\partial_t \vec{v} = \beta \vec{a} \qquad \partial_i \vec{v} = -\beta n_i \vec{a}$$

It follows that

$$\begin{aligned}
\partial_t(\vec{v} \cdot \vec{h}) &= (\partial_t \vec{v}) \cdot \vec{h} + \vec{v} \cdot (\partial_t \vec{h}) \\
&= (\beta \vec{a}) \cdot \vec{h} + \vec{v} \cdot (-\beta \vec{v}) \\
&= \beta h a_n - \beta v^2
\end{aligned}$$

$$\begin{aligned}
\partial_i(\vec{v} \cdot \vec{h}) &= (\partial_i \vec{v}) \cdot \vec{h} + \vec{v} \cdot (\partial_i \vec{h}) \\
&= (-\beta n_i \vec{a}) \cdot \vec{h} + \vec{v} \cdot (\hat{e}_i + \beta n_i \vec{v}) \\
&= -\beta h n_i a_n + v_i + \beta n_i v^2
\end{aligned}$$

Summary:

$$\partial_t(\vec{v} \cdot \vec{h}) = \beta h a_n - \beta v^2 \qquad \vec{\nabla}(\vec{v} \cdot \vec{h}) = -\beta h a_n \hat{n} + \vec{v} + \beta v^2 \hat{n}$$

And now,

$$\psi = \beta / h = ((1 - \vec{v} \cdot \hat{n}) h)^{-1} = (h - \vec{v} \cdot \vec{h})^{-1}$$

and so

$$\begin{aligned}
\partial_\mu \psi &= -\psi^2 \partial_\mu (\psi^{-1}) \\
&= -\psi^2 \partial_\mu (h - \vec{v} \cdot \vec{h}) \\
&= \psi^2 (\partial_\mu (\vec{v} \cdot \vec{h}) - \partial_\mu h)
\end{aligned}$$

$$\begin{aligned}
\partial_t \psi &= \psi^2 (\partial_t (\vec{v} \cdot \vec{h}) - \partial_t h) \\
&= \psi^2 (\beta h a_n - \beta v^2 - (1 - \beta))
\end{aligned}$$

$$\begin{aligned}
\vec{\nabla} \psi &= \psi^2 (\vec{\nabla}(\vec{v} \cdot \vec{h}) - \vec{\nabla} h) \\
&= \psi^2 (-\beta h a_n \hat{n} + \vec{v} + \beta v^2 \hat{n} - \beta \hat{n})
\end{aligned}$$

We are now in a position to calculate the fields, which we will carry out in the next section.

6.2 Fields due to point charge (completion)

We can write the scalar and vector potentials as follows (recall that we set $c = 1$, and so $\mu_0/(4\pi) = 1/(4\pi\epsilon_0)$):

$$\begin{aligned}
\phi &= \frac{q}{4\pi\epsilon_0} \psi \\
\vec{A} &= \frac{q}{4\pi\epsilon_0} \psi \vec{v}
\end{aligned}$$

If we define

$$\vec{e} = -\vec{\nabla}\psi - \partial_t(\psi\vec{v}) \tag{6.1}$$
$$\vec{b} = \vec{\nabla} \times (\psi\vec{v}) \tag{6.2}$$

then we find that

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \partial_t\vec{A} = \frac{q}{4\pi\epsilon_0}\vec{e} \\ \vec{B} &= \vec{\nabla} \times \vec{A} = \frac{q}{4\pi\epsilon_0}\vec{b} \end{aligned}$$

We start by calculating $\vec{e}$.

$$\begin{aligned} \partial_t(\psi\vec{v}) &= (\partial_t\psi)\vec{v} + \psi(\partial_t\vec{v}) \\ &= \psi^2(\beta h a_n - \beta v^2 - 1 + \beta)\vec{v} + \psi(\beta\vec{a}) \end{aligned}$$

$$\begin{aligned} \vec{e} &= -\vec{\nabla}\psi - \partial_t(\psi\vec{v}) \\ &= \psi^2(+\beta h a_n\hat{n} - \vec{v} - \beta v^2\hat{n} + \beta\hat{n}) + \psi^2(-\beta h a_n + \beta v^2 + 1 - \beta)\vec{v} - \psi(\beta\vec{a}) \\ &= \psi^2(\beta h a_n - \beta v^2 + \beta)(\hat{n} - \vec{v}) - \psi\beta\vec{a} \end{aligned}$$

Now calculate $\vec{b}$.

$$\begin{aligned} \partial_j(\psi v_k) &= (\partial_j\psi)v_k + \psi(\partial_j v_k) \\ &= \psi^2(-\beta h a_n n_j + v_j + \beta v^2 n_j - \beta n_j)v_k + \psi(-\beta n_j a_k) \end{aligned}$$

$$\begin{aligned} \vec{b} &= \vec{\nabla} \times (\psi\vec{v}) \\ &= \psi^2(-\beta h a_n\hat{n} + \vec{v} + \beta v^2\hat{n} - \beta\hat{n}) \times \vec{v} - \psi\beta\hat{n} \times \vec{a} \\ &= \psi^2(-\beta h a_n + \beta v^2 - \beta)\hat{n} \times \vec{v} - \psi\beta\hat{n} \times \vec{a} \end{aligned}$$

It seems that $\vec{e}$ and $\vec{b}$ have some interesting similarities. Let's calculate the following:

$$\begin{aligned} \hat{n} \times \vec{e} &= \hat{n} \times (\psi^2(\beta h a_n - \beta v^2 + \beta)(\hat{n} - \vec{v}) - \psi\beta\vec{a}) \\ &= \psi^2(\beta h a_n - \beta v^2 + \beta)\hat{n} \times (\hat{n} - \vec{v}) - \psi\beta\hat{n} \times \vec{a} \\ &= \psi^2(-\beta h a_n + \beta v^2 - \beta)\hat{n} \times \vec{v} - \psi\beta\hat{n} \times \vec{a} \\ &= \vec{b} \end{aligned}$$

Evidently, $\vec{b} = \hat{n} \times \vec{e}$.

Shifting our focus back to $\vec{e}$, we have found that

$$\vec{e} = \psi^2(\beta h a_n - \beta v^2 + \beta)(\hat{n} - \vec{v}) - \psi\beta\vec{a}$$

It turns out to be useful to split $\vec{e}$ into two contributions, and write $\vec{e} = \vec{e}_0 + \vec{e}_a$, where

$$\begin{aligned}\vec{e}_0 &= \psi^2(-\beta v^2 + \beta)(\hat{n} - \vec{v}) \\ &= \frac{\beta^3(1 - v^2)}{h^2}(\hat{n} - \vec{v})\end{aligned}$$

and

$$\begin{aligned}\vec{e}_a &= \psi^2(\beta h a_n)(\hat{n} - \vec{v}) - \psi\beta\vec{a} \\ &= \frac{\beta^3}{h^2} h a_n(\hat{n} - \vec{v}) - \frac{\beta^2}{h}\vec{a} \\ &= \frac{\beta^2(\beta a_n(\hat{n} - \vec{v}) - \vec{a})}{h}\end{aligned}$$

$\vec{e}_a$ represents contributions to the electric field that are due to the particle's acceleration, and $\vec{e}_0$ represents what the total electric field would be if the acceleration were zero.

Putting everything together, and reintroducing c into all of the equations (which can be done by dimensional analysis — $\vec{e}$ should have units of m^{-2} based on the electric field), we find

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{e} \qquad \vec{B} = \frac{1}{c}\hat{n} \times \vec{E} \tag{6.3}$$

$$\vec{e} = \vec{e}_0 + \vec{e}_a \tag{6.4}$$

$$\vec{e}_0 = \frac{\beta^3(1 - v(T)^2/c^2)}{|\vec{r} - \vec{R}(T)|^2}(\hat{n} - \vec{v}(T)/c) \tag{6.5}$$

$$\vec{e}_a = \frac{\beta^2}{c^2|\vec{r} - \vec{R}(T)|}(\beta(\hat{n} \cdot \vec{a}(T))(\hat{n} - \vec{v}(T)/c) - \vec{a}(T)) \tag{6.6}$$

$$T = t - |\vec{r} - \vec{R}(T)|/c \qquad \text{(retarded time)} \tag{6.7}$$

$$\hat{n} = \frac{\vec{r} - \vec{R}(T)}{|\vec{r} - \vec{R}(T)|} \qquad \text{(radial direction)} \tag{6.8}$$

$$\beta = (1 - \hat{n} \cdot \vec{v}(T)/c)^{-1} \tag{6.9}$$

Note that if $\vec{v}(T) = 0$ and $\vec{a}(T) = 0$, then $\vec{e}_a = 0$ and $\vec{e}_0 = \hat{n}/|\vec{r} - \vec{R}(T)|^2$, which is Coulomb's Law.

It is interesting to note that $\vec{e}_0$ is in the same direction as $\hat{n} - \vec{v}/c$, which is essentially radial for non-relativistic speeds (i.e., $|\vec{v}|/c \ll 1 = |\hat{n}|$). In fact, $\vec{e}_0$ turns out to be exactly radial (even at relativistic speeds) if we define "radial" using the correct reference point. More on this later.

By contrast,

$$\begin{aligned}
\hat{n}\cdot\vec{e}_a &= \hat{n}\cdot\frac{\beta^2}{h}(\beta a_n(\hat{n}-\vec{v})-\vec{a}) \\
&= \frac{\beta^2}{h}(\beta a_n(1-v_n)-a_n) \\
&= \frac{\beta^2 a_n}{h}(\beta(1-v_n)-1) \\
&= \frac{\beta^2 a_n}{h}(1-1) \\
&= 0
\end{aligned}$$

Evidently, the contribution to the electric field from the acceleration is perpendicular to the radial direction. Furthermore, the magnetic field is also perpendicular to the radial direction as well as to the electric field (see eq. 6.3). Furthermore, the magnitude of $\vec{e}_a$ drops off as $1/|\vec{r}-\vec{R}(T)|$. All of this is consistent with an electromagnetic wave being emitted by an accelerating charge.

By contrast, the electric field due to a non-accelerating charge is essentially radial (same direction as a hypothetical wave emitted from the source charge) and its magnitude drops off as $1/|\vec{r}-\vec{R}(T)|^2$, which is too rapid for a wave pulse carrying energy away to infinity in a conservative manner. Evidently, non-accelerating charges do *not* emit electromagnetic waves.

6.3 Electric field due to a charge moving at constant velocity

...our prediction (based on e_0)...

...alternative prediction based on relativistic transformation from charges rest frame to moving frame...

...relativity formula calculates E in terms of $R(t)$, the position of the particle at present time. appears to violate causality (speed-of-light limit)...

...we can equate the two predictions (write relativity version in terms of $\vec{e}'$)...

...we can go one step further, and come up with an alternative to e_0 (that will work in general) which is based on relativity version...

...key to doing that is to replace $R(t)$ with R_0, which is a "predicted" location of charge at t based on $R(T)$, $v(T)$, and an assumption that there is no acceleration during $[T,t]$ time interval...

...to this end, define R_0.....

...the fields at $(\vec{r},t)$ cannot depend on what the charge is actually doing after the retarded time...

6.4 Alternative formula for electric field

Alternative formula (used by Feynmann in his Lecture Notes, see Vol. I, Ch. 28 and Vol. II, Ch. 21):

$$\vec{e}' = \frac{\hat{n}}{h^2} + \frac{h}{c}\partial_t(\frac{\hat{n}}{h^2}) + \frac{1}{c^2}\partial_t^2\hat{n} \tag{6.10}$$

Label each term $\vec{e}'_1$, $\vec{e}'_2$, and $\vec{e}'_3$, respectively.

Our goal in this section is to show that $\vec{e}' = \vec{e}$ as we derived in Section 06-02. Rather than rederive all the terms that make up $\vec{e}$ as we evaluate the time derivatives in $\vec{e}'$, we will try to focus on all of the terms that will eventually cancel out.

Thus, it makes sense to recall the formulas for the derivatives of ψ and to further manipulate them:

$$\begin{aligned}\partial_t\psi &= \psi^2(\beta h a_n - \beta v^2 - (1-\beta)) \\ &= \psi^2(\beta h a_n + \beta(1-v^2) - 1)\end{aligned}$$

This leads to

$$\begin{aligned}\vec{\nabla}\psi &= \psi^2(-\beta h a_n\hat{n} + \vec{v} + \beta v^2\hat{n} - \beta\hat{n}) \\ &= \psi^2(-\beta h a_n\hat{n} + \hat{n} + \beta v^2\hat{n} - \beta\hat{n}) - \psi^2(\hat{n} - \vec{v}) \\ &= \psi^2(-\beta h a_n\hat{n} - \beta(1-v^2)\hat{n} + \hat{n}) - \psi^2(\hat{n} - \vec{v}) \\ &= -(\partial_t\psi)\hat{n} - \psi^2(\hat{n} - \vec{v})\end{aligned}$$

which leads to

$$\begin{aligned}\vec{e} &= -\vec{\nabla}\psi - \partial_t(\psi\vec{v}) \\ &= (\partial_t\psi)\hat{n} - \partial_t(\psi\vec{v}) + \psi^2(\hat{n} - \vec{v})\end{aligned}$$

and so

$$(\partial_t\psi)\hat{n} - \partial_t(\psi\vec{v}) = \vec{e} - \psi^2(\hat{n} - \vec{v})$$

We will also need the following (recall that $\partial_t\vec{h} = -\beta\vec{v}$ and $\partial_t h = 1 - \beta$):

$$\begin{aligned}\partial_t(\frac{\vec{h}}{h^k}) &= \partial_t(\vec{h}h^{-k}) \\ &= (\partial_t\vec{h})h^{-k} + \vec{h}(-kh^{-k-1})(\partial_t h) \\ &= (-\beta\vec{v})h^{-k} - k\vec{h}h^{-(k+1)}(1-\beta) \\ &= -\frac{1}{h^k}(\beta\vec{v} + k(1-\beta)\hat{n})\end{aligned}$$

And now we evaluate each term in $\vec{e}'$:

$$\vec{e}'_1 = \frac{\hat{n}}{h^2}$$

$$\begin{aligned}\vec{e}'_2 &= h\partial_t(\frac{\hat{n}}{h^2}) \\ &= h\partial_t(\frac{\vec{h}}{h^3}) \\ &= -h\frac{1}{h^3}(\beta\vec{v} + 3(1-\beta)\hat{n}) \\ &= -\frac{1}{h^2}(\beta\vec{v} + 3(1-\beta)\hat{n})\end{aligned}$$

$$\begin{aligned}\partial_t\hat{n} &= \partial_t(\frac{\vec{h}}{h^1}) \\ &= -\frac{1}{h}(\beta\vec{v} + (1-\beta)\hat{n}) \\ &= -\psi\vec{v} + \psi\hat{n} - \frac{\vec{h}}{h^2}\end{aligned}$$

$$\begin{aligned}\vec{e}'_3 &= \partial_t^2\hat{n} \\ &= \partial_t(-\psi\vec{v} + \psi\hat{n} - \frac{\vec{h}}{h^2}) \\ &= -\partial_t(\psi\vec{v}) + (\partial_t\psi)\hat{n} + \psi\partial_t\hat{n} - \partial_t(\frac{\vec{h}}{h^2}) \\ &= \vec{e} - \psi^2(\hat{n} - \vec{v}) + \psi(-\psi\vec{v} + \psi\hat{n} - \frac{\vec{h}}{h^2}) + \frac{1}{h^2}(\beta\vec{v} + 2(1-\beta)\hat{n}) \\ &= \vec{e} - \psi^2(\hat{n} - \vec{v}) + \psi^2(\hat{n} - \vec{v}) - \psi\frac{\hat{n}}{h} + \frac{\beta}{h^2}\vec{v} + 2(1-\beta)\frac{\hat{n}}{h^2} \\ &= \vec{e} - \frac{\beta}{h^2}\hat{n} + \frac{\beta}{h^2}\vec{v} + 2(1-\beta)\frac{\hat{n}}{h^2}\end{aligned}$$

Putting it all together gives us

$$\begin{aligned}\vec{e}' &= \vec{e}'_1 + \vec{e}'_2 + \vec{e}'_3 \\ &= \frac{\hat{n}}{h^2} - \frac{\beta}{h^2}\vec{v} - 3(1-\beta)\frac{\hat{n}}{h^2} + \vec{e} - \frac{\beta}{h^2}\hat{n} + \frac{\beta}{h^2}\vec{v} + 2(1-\beta)\frac{\hat{n}}{h^2} \\ &= \vec{e} + \frac{\hat{n}}{h^2} - \frac{\beta}{h^2}\hat{n} - (1-\beta)\frac{\hat{n}}{h^2} \\ &= \vec{e} + (1-\beta)\frac{\hat{n}}{h^2} - (1-\beta)\frac{\hat{n}}{h^2} \\ &= \vec{e}\end{aligned}$$

7 Local Charge Motion

7.1 Local charge motion

Recall the electric field, which is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{e}$$

where

$$\vec{e} = \frac{\beta^3(1-v^2)}{h^2}(\hat{n}-\vec{v}) + \frac{\beta^2(\beta a_n(\hat{n}-\vec{v})-\vec{a})}{h}$$

In this section, we will perform a first order power series expansion in $\vec{R}(T)$ and $\vec{v}(T)$. This turns out to be appropriate whenever $|\vec{R}(t')| \ll r$ and $|\vec{v}(t')| \ll c$ for all times t'. We will not be assuming that $\vec{a}(t')$ is small, although it turns out to be largely unnecessary, as $\vec{e}$ is already linear in $\vec{a}$.

We will begin with $|\vec{h}|$ ($\vec{h} = \vec{r} - \vec{R}$ is already first order in $\vec{R}$):

$$\begin{aligned} h^2 &= (\vec{r}-\vec{R})\cdot(\vec{r}-\vec{R}) \\ &= \vec{r}\cdot\vec{r} - 2\vec{r}\cdot\vec{R} + \vec{R}\cdot\vec{R} \\ &\approx r^2 - 2r\hat{r}\cdot\vec{R} \\ &= r^2(1-2R_r/r) \end{aligned}$$

It follows that

$$\begin{aligned} h^n &= (h^2)^{n/2} \\ &\approx (r^2(1-2R_r/r))^{n/2} \\ &\approx r^n(1-(n/2)2R_r/r) \qquad \text{(binomial expansion)} \\ &= r^n(1-nR_r/r) \end{aligned}$$

And so,

$$\begin{aligned} \hat{n} &= \vec{h}/h \\ &\approx (\vec{r}-\vec{R})(r^{-1})(1+R_r/r) \\ &= (\hat{r}-\vec{R}/r)(1+R_r/r) \\ &= \hat{r} + ((R_r/r)\hat{r} - \vec{R}/r) \end{aligned}$$

It follows that

$$\begin{aligned} v_n &= \vec{v}\cdot\hat{n} \\ &= \vec{v}\cdot(\hat{r} + ((R_r/r)\hat{r} - \vec{R}/r)) \\ &= v_r + ((R_r/r)v_r - (\vec{v}\cdot\vec{R})/r) \end{aligned}$$

Given that $\vec{v}$ and $\vec{R}$ are both small, it makes sense to discard the product terms, and just keep the first term:

$$v_n = v_r$$

However, since we are *not* assuming that $\vec{a}$ is small, it is appropriate to keep those product terms, at least for now, when calculating a_n (pretty much the same calculation):

$$a_n = a_r + ((R_r/r)a_r - (\vec{a}\cdot\vec{R})/r)$$

We have

$$\beta^n = (1 - v_n)^{-n} = (1 - v_r)^{-n} = 1 + nv_r$$

and so

$$\begin{aligned}
\frac{\beta^3(1-v^2)}{h^2} &\approx \frac{\beta^3}{h^2} \\
&= (1+3v_r)(r^{-2})(1+2R_r/r) \\
&= \frac{1}{r^2}(1+2R_r/r+3v_r)
\end{aligned}$$

It follows that

$$\begin{aligned}
\vec{e}_0 &= \frac{\beta^3(1-v^2)}{h^2}(\hat{n}-\vec{v}) \\
&= \frac{1}{r^2}(1+2R_r/r+3v_r)(\hat{r}+(R_r/r)\hat{r}-\vec{R}/r-\vec{v}) \\
&= \frac{1}{r^2}(\hat{r}+(2R_r/r+3v_r)\hat{r}+((R_r/r)\hat{r}-\vec{R}/r-\vec{v})) \\
&= \frac{1}{r^2}(\hat{r}+(3(R_r/r)\hat{r}-\vec{R}/r)+(3v_r\hat{r}-\vec{v})) \\
&= \frac{1}{r^2}(\hat{r}+(3\hat{r}\otimes\hat{r}-I)\vec{R}/r+(3\hat{r}\otimes\hat{r}-I)\vec{v}
\end{aligned}$$

The acceleration terms are a little more complicated. We start with

$$\begin{aligned}
\frac{\beta^3}{h}(\hat{n}-\vec{v}) &= (1+3v_r)(r^{-1})(1+R_r/r)(\hat{r}+(R_r/r)\hat{r}-\vec{R}/r+\vec{v}) \\
&= \frac{1}{r}(1+3v_r+R_r/r)(\hat{r}+(R_r/r)\hat{r}-\vec{R}/r+\vec{v}) \\
&= \frac{1}{r}(\hat{r}+(3v_r+R_r/r)\hat{r}+((R_r/r)\hat{r}-\vec{R}/r+\vec{v})) \\
&= \frac{1}{r}(\hat{r}+(2(R_r/r)\hat{r}-\vec{R}/r)+(3v_r\hat{r}-\vec{v}))
\end{aligned}$$

$$\begin{aligned}
\vec{e}_{a1} &= \frac{\beta^3 a_n(\hat{n}-\vec{v})}{h} \\
&= \frac{1}{r}(\hat{r}+(2(R_r/r)\hat{r}-\vec{R}/r)+(3v_r\hat{r}-\vec{v}))(a_r+((R_r/r)a_r-(\vec{a}\cdot\vec{R})/r)) \\
&= \frac{1}{r}(a_r\hat{r}+a_r(2(R_r/r)\hat{r}-\vec{R}/r)+a_r(3v_r\hat{r}-\vec{v})+((R_r/r)a_r-(\vec{a}\cdot\vec{R})/r)\hat{r}) \\
&= \frac{1}{r}(a_r\hat{r}+a_r(3(R_r/r)\hat{r}-\vec{R}/r)-(\vec{a}\cdot\vec{R}/r)\hat{r}+a_r(3v_r\hat{r}-\vec{v})) \\
&= \frac{1}{r}(a_r\hat{r}+a_r(3\hat{r}\otimes\hat{r}-I)\vec{R}/r-(\vec{a}\cdot\vec{R}/r)\hat{r}+a_r(3\hat{r}\otimes\hat{r}-I)\vec{v})
\end{aligned}$$

$$
\begin{aligned}
\vec{e}_{a2} &= -\frac{\beta^2\vec{a}}{h} \\
&= -(1+2v_r)(r^{-1})(1+R_r/r)\vec{a} \\
&= -\frac{1}{r}(1+2v_r+R_r/r)\vec{a}
\end{aligned}
$$

Adding $\vec{e}_0$, $\vec{e}_{a1}$ and $\vec{e}_{a2}$ together, and putting c back in, yields the following:

$$
\begin{aligned}
\vec{e} &= \frac{\hat{r}}{r^2} \qquad (\vec{e}\,'_0) \\
&\quad +\frac{1}{r^3}(3\hat{r}\otimes\hat{r}-I)\vec{R} \qquad (\vec{e}\,'_R) \\
&\quad +\frac{1}{cr^2}(3\hat{r}\otimes\hat{r}-I)\vec{v} \qquad (\vec{e}\,'_v) \\
&\quad +\frac{1}{c^2r}(\hat{r}\otimes\hat{r}-I)\vec{a} \qquad (\vec{e}\,'_a) \\
&\quad +\frac{1}{c^2r^2}(a_r(3\hat{r}\otimes\hat{r}-I)\vec{R}-(\vec{a}\cdot\vec{R})\hat{r}-R_r\vec{a}) \qquad (\vec{e}\,'_{aR}) \\
&\quad +\frac{1}{c^3r}(a_r(3\hat{r}\otimes\hat{r}-I)\vec{v}-2v_r\vec{a}) \qquad (\vec{e}\,'_{av})
\end{aligned}
$$

The different contributions to $\vec{e}$ have been labelled to the right. The term $\vec{e}\,'_0$ represents the leading order term (where $\vec{R}=\vec{v}=\vec{a}=0$), and gives rise to Coulomb's Law. Both $\vec{e}\,'_R$ and $\vec{e}\,'_v$ are subleading terms when compared to $\vec{e}\,'_0$, and represent the portions of $\vec{e}$ that have been expanded to first order in $\vec{R}$ and $\vec{v}$, respectively, and do not depend on acceleration.

The term $\vec{e}\,'_a$ is the leading order term involving acceleration (with no dependence on $\vec{R}$ or $\vec{v}$). The terms $\vec{e}\,'_{aR}$ and $\vec{e}\,'_{av}$ are subleading to $\vec{e}\,'_a$ and represent portions of $\vec{e}$ which depend on $\vec{a}$ and have been expanded to first order in $\vec{R}$ and $\vec{v}$, respectively.

Based on our underlying assumption that $|\vec{R}(t')| \ll r$ and $|\vec{v}(t')| \ll c$ for all t', it follows that $\vec{e}\,'_R$ and $\vec{e}\,'_v$ are both much less than the Coulomb term $\vec{e}\,'_0$, and that $\vec{e}\,'_{aR}$ and $\vec{e}\,'_{av}$ are both much less than $\vec{e}\,'_a$. Due to the latter observation, it is normal to discard $\vec{e}\,'_{aR}$ and $\vec{e}\,'_{av}$, and we shall do so without further discussion.

One might think we should also discard $\vec{e}\,'_R$ and $\vec{e}\,'_v$ as well, since they are much less than the Coulomb term $\vec{e}\,'_0$. However, it may not be appropriate to do so. If, in addition to a charge q wandering around locally according to the position function $\vec{R}(t')$, there is a charge $-q$ fixed at the origin (i.e., a time-varying electric dipole), the Coulomb terms from the q and $-q$ charges would cancel out, leaving us with the remaining terms for the q charge.

The comparison between $\vec{e}\,'_a$ and the other terms is less clear, since we are not necessarily assuming the $\vec{a}$ is small in any particular manner. If we assume that the motion of the charge is governed by a time-scale parameter τ in the sense that $|\vec{v}| \sim (1/\tau)|\vec{R}|$ and $|\vec{a}| \sim (1/\tau)|\vec{v}|$, then the comparison between $\vec{e}\,'_R$, $\vec{e}\,'_v$, and $\vec{e}\,'_a$ will depend on the comparison of τ to r/c.

For example, if $\tau \gg r/c$ (e.g., low frequeny oscillations) then the $\vec{e}\,'_R$ term will dominate over $\vec{e}\,'_v$ and $\vec{e}\,'_a$. Likewise, if $\tau \ll r/c$ (e.g., high frequency oscillations) then the $\vec{e}\,'_a$ term will

dominate over $\vec{e}'_R$ and $\vec{e}'_v$. If τ and r/c are similar in value, then it is quite likely that all three terms ($\vec{e}'_R$, $\vec{e}'_v$ and $\vec{e}'_a$) are of equal importance.

In fact, in the second case where $\tau \ll r/c$, it is quite possible that $\vec{e}'_a$ will even dominate over the Coulomb term (i.e., if $|\vec{a}|r \gg c^2$). It should be noted that $\vec{e}'_a$ is the only term that varies as $1/r$ (all other terms drop as $1/r^2$ or faster), and is thus the only term that can represent an EM wave carrying energy away from the dipole.

7.2 Local charge motion (magnetic field)

Recall that

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{e}$$

where $\vec{e}$ is given to first order in $\vec{R}$ and $\vec{v}$ by

$$\vec{e} = \vec{e}'_0 + \vec{e}'_R + \vec{e}'_v + \vec{e}'_a$$

where each of the terms is given in the last section. This first order expansion is appropriate when $|\vec{R}(t')| \ll r$ and $|\vec{v}(t')| \ll c$ for all t'.

The magnetic field can then be calculated via

$$\vec{B} = \frac{q}{4\pi\epsilon_0 c}\vec{b}$$

where $\vec{b} = \hat{n} \times \vec{e}$, with $\vec{e}$ given as above.

To carry out the expansion of $\vec{b}$ to first order in $\vec{R}$ and $\vec{v}$, we first recall that

$$\hat{n} = \hat{r} + \vec{n}_1$$

where

$$\vec{n}_1 = (R_r/r)\hat{r} - \vec{R}/r = \frac{1}{r}(\hat{r} \otimes \hat{r} - I)\vec{R}$$

Note that, since $\vec{e}'_R$ and $\vec{e}'_v$ are already first order in $\vec{R}$ and $\vec{v}$, respectively, and that we have discarded terms that are first order in $\vec{R}$ and $\vec{a}$ when we dropped $\vec{e}'_{aR}$, we only need to include $\vec{n}_1$ when expanding the $\hat{n} \times \vec{e}'_0$ term.

It should be noted during the calculation that

$$((\hat{r} \otimes \hat{r})\vec{w}) \times \hat{r} = (\hat{r} \cdot \vec{w})\hat{r} \times \hat{r} = 0$$

for any vector $\vec{w}$.

Since magnetic fields are not produced by stationary charges, regardless of where they are, it should come as no surprise that the following two terms cancel each other completely:

$$\begin{aligned}
\hat{n} \times \vec{e}'_0 &= (\hat{r} + \vec{n}_1) \times \frac{1}{r^2}\hat{r} \\
&= \frac{1}{r^3}((\hat{r} \otimes \hat{r} - I)\vec{R}) \times \hat{r} \\
&= -\frac{1}{r^3}\vec{R} \times \hat{r} \\
&= \frac{1}{r^3}\hat{r} \times \vec{R}
\end{aligned}$$

$$
\begin{aligned}
\hat{n} \times \vec{e}\,'_R &= \hat{r} \times \frac{1}{r^3}(3\hat{r} \otimes \hat{r} - I)\vec{R} \\
&= -\frac{1}{r^3}\hat{r} \times \vec{R}
\end{aligned}
$$

The remaining two terms are easily calculated:

$$
\begin{aligned}
\vec{b}\,'_v &= \hat{n} \times \vec{e}\,'_v \\
&= \hat{r} \times \frac{1}{cr^2}(3\hat{r} \otimes \hat{r} - I)\vec{v} \\
&= -\frac{1}{cr^2}\hat{r} \times \vec{v} \\
&= \frac{1}{cr^2}\vec{v} \times \hat{r}
\end{aligned}
$$

$$
\begin{aligned}
\vec{b}\,'_a &= \hat{n} \times \vec{e}\,'_a \\
&= \hat{r} \times \frac{1}{c^2 r}(3\hat{r} \otimes \hat{r} - I)\vec{a} \\
&= -\frac{1}{c^2 r}\hat{r} \times \vec{a} \\
&= \frac{1}{c^2 r}\vec{a} \times \hat{r}
\end{aligned}
$$

It follows that

$$
\vec{b} = \vec{b}\,'_v + \vec{b}\,'_a = \frac{1}{cr^2}\vec{v} \times \hat{r} + \frac{1}{c^2 r}\vec{a} \times \hat{r}
$$

7.3 Local charge motion (retarded time)

In previous two sections, $\vec{E}$ and $\vec{B}$ are calculated to first order in $\vec{R}(T)$ and $\vec{v}(T)$ under the assumption that $|\vec{R}(t')| \ll r$ and $|\vec{v}(t')| \ll c$ for all t'. The functions $\vec{R}$, $\vec{v}$ and $\vec{a}$ are to be evaluated at the retarded time, T, which is determined implicitly by solving the following equation:

$$
T = t - |\vec{r} - \vec{R}(T)|/c \tag{7.1}
$$

This leads to an obvious question, is there an appropriate first order expansion of T based on the smallness of $\vec{R}$ and $\vec{v}$? This question is somewhat tricky due to the implicit definition given above, and the fact that the functions $\vec{R}$ and $\vec{v}$ can change in non-obvious ways when their argument, T, is shifted (recall that $\vec{a}$, which is the derivative of $\vec{v}$, is *not* assumed to be small).

It seems that the leading order value of T can be easily found simply by setting $\vec{R}$ (and $\vec{v}$) to zero. In that case,

$$
T_0 = t - |\vec{r} - \vec{0}|/c = t - r/c
$$

One way to try to get a first order correction to T is to plug T_0 into the RHS of the implicit definition above

$$
T_1 = t - |\vec{r} - \vec{R}(T_0)|/c
$$

and then expand this to first order in $\vec{R}(T_0)$:

$$\begin{aligned}
T_1 &= t - |\vec{r} - \vec{R}(T_0)|/c \\
&= t - (1/c)\sqrt{(\vec{r} - \vec{R}(T_0)) \cdot (\vec{r} - \vec{R}(T_0))} \\
&= t - (1/c)\sqrt{r^2 - 2\vec{r} \cdot \vec{R}(T_0) + R(T_0)^2} \\
&\approx t - (r/c)\sqrt{1 - 2\hat{r} \cdot \vec{R}(T_0)/r} \\
&= t - (r/c)(1 - R_r(T_0)/r) \\
&= t - r/c + R_r(T_0)/c
\end{aligned}$$

Of course, T_1 as defined above is not the same as T because the latter requires the $\vec{R}$ to be evaluated at T itself, rather than T_0, so it may be premature to assert that the first order expansion of T_1 is a valid first order expansion of T. However, with the help of the fact that $\vec{v}$ is also small, we can actually construct an iron-clad calculation of T expanded to first order in $\vec{R}$ and $\vec{v}$, and it turns out to be the expansion of T_1, as given above.

Let us begin by constructing a recursive algorithm for approximating T which can be proven to converge to T itself. It is based on the following assumption, that there exists a value of $\alpha < 1$ such that $|\vec{v}(t')| \leq \alpha c$ for all t'. Of course, with $\vec{v}$ assumed to be small compared to c, we would expect $\alpha \ll 1$, but we don't need to assume that $\vec{v}$ is that small in order to justify the recursive algorithm. We simply need to assume that the particle travels at a speed less than c, and does not approach c arbitrarily closely during its entire trajectory.

We now define the sequence $(T_0, T_1, T_2, \ldots, T_n, \ldots)$ by

$$T_0 = t - r/c \tag{7.2}$$

$$T_{n+1} = t - |\vec{r} - \vec{R}(T_n)|/c \qquad n = 0, 1, 2, \ldots \tag{7.3}$$

It then follows for all such n that

$$\begin{aligned}
|T_{n+2} - T_{n+1}| &= |(t - |\vec{r} - \vec{R}(T_{n+1})|/c) - (t - |\vec{r} - \vec{R}(T_n)|/c)| \\
&= ||\vec{r} - \vec{R}(T_{n+1})|/c - |\vec{r} - \vec{R}(T_n)|/c| \\
&\leq |\vec{R}(T_{n+1}) - \vec{R}(T_n)|/c \\
&= \frac{1}{c}|\int_{T_n}^{T_{n+1}} \vec{v}(t')\, dt'| \\
&\leq \frac{1}{c}\int_{T_n}^{T_{n+1}} |\vec{v}(t')|\, |dt'| \\
&\leq \frac{1}{c}\int_{T_n}^{T_{n+1}} \alpha c\, |dt'| \\
&= \alpha \int_{T_n}^{T_{n+1}} |dt'| \\
&= \alpha |T_{n+1} - T_n|
\end{aligned}$$

It now follows by an inductive argument that

$$|T_{n+1} - T_n| \leq \alpha^n |T_1 - T_0|$$

and so for all $m, n = 0, 1, 2, \ldots$ with $m > n$:

$$\begin{aligned} |T_m - T_n| &\leq \sum_{k=n}^{m-1} |T_{k+1} - T_k| \\ &\leq \sum_{k=n}^{m-1} \alpha^k |T_1 - T_0| \\ &= \alpha^n \sum_{k=0}^{m-n-1} \alpha^k |T_1 - T_0| \\ &\leq \alpha^n \sum_{k=0}^{\infty} \alpha^k |T_1 - T_0| \\ &= \frac{\alpha^n}{1-\alpha} |T_1 - T_0| \end{aligned}$$

It thus follows that the sequence $(T_0, T_1, T_2, \ldots)$ is Cauchy, and therefore convergent (completeness of the reals is needed here), and thus we may define

$$T = \lim_{n \to \infty} T_n$$

If we take the limit $n \to \infty$ with eq. 7.2, we obtain

$$T = t - |\vec{r} - \vec{R}(T)|/c$$

which implies that T, as defined by the limit above, is the retarded time.[2]

Furthermore, we find that

$$|T - T_n| = \lim_{m \to \infty} |T_m - T_n| \leq \frac{\alpha^n}{1-\alpha} |T_1 - T_0|$$

Since $|T_1 - T_0|$ is already first order in $\vec{R}$, it turns out that a first order expansion of T in $\vec{R}$ and $\vec{v}$ (with product terms dropped) is given by a first order expansion of T_1 (i.e., all terms in $|T - T_1|$ are to be dropped at this order), and so

$$T \approx T_1 \approx t - r/c + R_r(t - r/c)/c \tag{7.4}$$

[2] This result is actually a special case of the "Contraction Fixed Point Theorem", which states that any function f satisfying $|f(t) - f(t')| \leq \alpha |t - t'|$ for all t and t' for some $\alpha < 1$ possesses a unique fixed point t satisfying $f(t) = t$. This Theorem, which can be generalized to functions mapping any complete normed vector space to itself, plays a pivotal role in proving the existance and uniqueness of solutions to a broad category of initial value differential equations.

7.4 Dipole radiation

We will now consider the situation where a charge q is oscillating around the origin along the z-axis, and a $-q$ charge is fixed at the origin — the so-called "oscillating dipole". The motion of the charge q is given by

$$\vec{R}(t') = (z_m \hat{z}) \cos(\omega t')$$

where ω is the angular frequency and $z_m \hat{z}$ is the amplitude of the oscillation. The dipole amplitude is given by

$$\vec{p}_m = p_m \hat{z} = q z_m \hat{z}$$

As is normal practice, we will be using complex phasors to represent the oscillation. Thus, we will write

$$\vec{R}(t') = (z_m \hat{z}) e^{-i\omega t'}$$

with the understanding that the real part of $\vec{R}(t')$ represents the actual motion of the charge.

Any quantity that is a linear function of $\vec{R}$ can, likewise, be represented by a phasor whose real part represents the actual value of that quantity. Thus, it follows that the velocity and acceleration of the charge is given by

$$\begin{aligned}
\vec{v}(t') &= \frac{d\vec{R}}{dt'} = (-i\omega z_m \hat{z}) e^{-i\omega t'} \\
\vec{a}(t') &= \frac{d\vec{v}}{dt'} = (-\omega^2 z_m \hat{z}) e^{-i\omega t'}
\end{aligned}$$

We will assume that the dipole amplitude is small, in the sense z_m is much less than the wavelength of the emitted electromagnetic wave as well as the distance from the origin to the field point. This is a reasonable approximation, since z_m is usually on the order of the atomic scale (0.1 nm), and wavelengths of visible light are in the 500 nm range (the approximation breaks down in the soft X-ray range). Likewise, we are almost always concerned with field points that are much farther away than atomic scale, and in fact, are often at macroscopic distances (centimeters and beyond).

Mathematically, these assumptions can be expressed as follows:

$$\begin{aligned}
z_m &\ll r \\
z_m &\ll \lambda/(2\pi) = 1/k = c/\omega
\end{aligned}$$

from which it follows that

$$\begin{aligned}
|\vec{R}(t')| &\leq z_m \ll r \\
|\vec{v}(t')| &\leq \omega z_m \ll c
\end{aligned}$$

We may therefore use the first order approximation that we derived in the previous sections for $\vec{E}$ and $\vec{B}$:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{e} \qquad \vec{B} = \frac{q}{4\pi\epsilon_0 c}\vec{b}$$

where

$$\vec{e} = \frac{1}{r^3}(3\hat{r}\otimes\hat{r} - I)\vec{R} + \frac{1}{cr^2}(3\hat{r}\otimes\hat{r} - I)\vec{v} + \frac{1}{c^2 r}(\hat{r}\otimes\hat{r} - I)\vec{a}$$

and

$$\vec{b} = \frac{1}{cr^2}\vec{v}\times\hat{r} + \frac{1}{c^2 r}\vec{a}\times\hat{r}$$

Note that we omitted the Coulomb term for the electric field due to the q charge because that term is cancelled by the electric field due to the $-q$ charge fixed at the origin. The $-q$ charge does not contribute to the magnetic field at all.

It is understood that $\vec{R}$, $\vec{v}$ and $\vec{a}$ should be evaluated at the retarded time T, given to first order by

$$T = t - r/c + R_r(t - r/c)/c$$

The phasor part is then given by

$$e^{-i\omega T} = e^{i\Phi}$$

where

$$\begin{aligned}
\Phi &= -\omega T \\
&= -\omega(t - r/c + R_r(t - r/c)/c) \\
&= -\omega t + (\omega/c)(r - R_r(t - r/c)) \\
&= kr - \omega t - (\omega/c)R_r(t - r/c)
\end{aligned}$$

Since $z_m\omega \ll c$, the first-order correction term in Φ much less than 1 rad, and will therefore have little effect on the phase of the out-going wave. However, the correction term is time-dependent, and will therefore affect the observed frequency of the wave. The adjusted frequency will be given by

$$\omega' = -\frac{d\Phi}{dt} = \omega + (\omega/c)v_r(t - r/c) = \omega(1 + v_r(t - r/c)/c)$$

The adjustment can be attributed entirely to the Doppler shift. Since $|\vec{v}| \ll c$, we usually drop this term altogether, and thus the phasor part will be given by

$$e^{-i\omega T} = e^{i(kr - \omega t)}$$

which is consistent with a spherical wave propagating outward from the origin with speed $\omega/k = c$.

We will be using spherical coordinates (r, θ, ϕ) to express the field point, noting that

$$\begin{aligned}
\hat{r} &= \cos\theta\hat{z} + \sin\theta\hat{\rho} \\
\hat{\theta} &= -\sin\theta\hat{z} + \cos\theta\hat{\rho} \\
\hat{\rho} &= \cos\phi\hat{x} + \sin\phi\hat{y} \\
\hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y} = \hat{z}\times\hat{\rho} = \hat{r}\times\hat{\theta}
\end{aligned}$$

It follows that

$$\begin{aligned}(\hat{r}\otimes\hat{r})\hat{z} &= \hat{r}(\hat{r}\cdot\hat{z}) \\ &= \cos\theta\hat{r}\end{aligned}$$

and

$$\begin{aligned}(\hat{r}\otimes\hat{r}-I)\hat{z} &= \cos\theta\hat{r}-\hat{z} \\ &= \cos\theta(\cos\theta\hat{z}+\sin\theta\hat{\rho})-\hat{z} \\ &= (\cos^2\theta-1)\hat{z}+\cos\theta\sin\theta\hat{\rho} \\ &= -\sin^2\theta\hat{z}+\cos\theta\sin\theta\hat{\rho} \\ &= \sin\theta(-\sin\theta\hat{z}+\cos\theta\hat{\rho}) \\ &= \sin\theta\hat{\theta}\end{aligned}$$

It follows that

$$\begin{aligned}(3\hat{r}\otimes\hat{r}-I)\hat{z} &= (\hat{r}\otimes\hat{r}-I)\hat{z}+2(\hat{r}\otimes\hat{r})\hat{z} \\ &= \sin\theta\hat{\theta}+2\cos\theta\hat{r}\end{aligned}$$

We also have

$$\begin{aligned}\hat{z}\times\hat{r} &= \hat{z}\times(\cos\theta\hat{z}+\sin\theta\hat{\rho}) \\ &= \sin\theta(\hat{z}\times\hat{\rho}) \\ &= \sin\theta\hat{\phi}\end{aligned}$$

If we define $\eta = c/(r\omega) = 1/(kr)$, then

$$\begin{aligned}\vec{e} &= \frac{1}{r^3}(3\hat{r}\otimes\hat{r}-I)\vec{R}+\frac{1}{cr^2}(3\hat{r}\otimes\hat{r}-I)\vec{v}+\frac{1}{c^2r}(\hat{r}\otimes\hat{r}-I)\vec{a} \\ &= \frac{z_m e^{-i\omega T}}{r^3}(\sin\theta\hat{\theta}+2\cos\theta\hat{r})+\frac{-i\omega z_m e^{-i\omega T}}{cr^2}(\sin\theta\hat{\theta}+2\cos\theta\hat{r})+\frac{-\omega^2 z_m e^{-i\omega T}}{c^2r}(\sin\theta\hat{\theta}) \\ &= -\frac{\omega^2 z_m}{c^2r}(\sin\theta(1+i\eta-\eta^2)\hat{\theta}+2\cos\theta(i\eta-\eta^2)\hat{r})e^{i(kr-\omega t)}\end{aligned}$$

and

$$\begin{aligned}\vec{b} &= \frac{1}{cr^2}\vec{v}\times\hat{r}+\frac{1}{c^2r}\vec{a}\times\hat{r} \\ &= \frac{-i\omega z_m e^{-i\omega T}}{cr^2}(\sin\theta\hat{\phi})+\frac{-\omega^2 z_m e^{-i\omega T}}{c^2r}(\sin\theta\hat{\phi}) \\ &= -\frac{\omega^2 z_m}{c^2r}(\sin\theta(1+i\eta)\hat{\phi})e^{i(kr-\omega t)}\end{aligned}$$

It follows that the electric and magnetic fields themselves are given by

$$\vec{E} = -\frac{\omega^2 p_m}{4\pi\epsilon_0 c^2 r}(\sin\theta(1+i\eta-\eta^2)\hat{\theta}+2\cos\theta(i\eta-\eta^2)\hat{r})e^{i(kr-\omega t)} \tag{7.5}$$

$$\vec{B} = -\frac{\omega^2 p_m}{4\pi\epsilon_0 c^3 r}(\sin\theta(1+i\eta)\hat{\phi})e^{i(kr-\omega t)} \tag{7.6}$$

Eqs. 7.5 and 7.6 are valid whenever $z_m \ll r$ and $z_m \ll c/\omega = 1/k = \lambda/(2\pi)$.

It should be noted that, up to this point, we have made no assumptions about the comparison between r and $\lambda/(2\pi)$. Under the additional assumption that $\lambda/(2\pi) \ll r$, so that

$$z_m \ll c/\omega = 1/k = \lambda/(2\pi) \ll r$$

then $\eta \ll 1$, and thus η can be set to zero in eqs. 7.5 and 7.6. Note that the acceleration terms in the electric and magnetic field dominate over all other terms in this case.

The result is the following

$$\vec{E} = \frac{\omega^2 p_m \sin\theta}{4\pi\epsilon_0 c^2 r}(-\hat{\theta})e^{i(kr-\omega t)} \tag{7.7}$$

$$\vec{B} = \frac{\omega^2 p_m \sin\theta}{4\pi\epsilon_0 c^3 r}(-\hat{\phi})e^{i(kr-\omega t)} \tag{7.8}$$

Eqs. 7.7 and 7.8 are more commonly reported in the literature. Note that they represent spherical electromagnetic waves propagating outwards from the origin, with $\vec{E}$, $\vec{B}$, and $\hat{r}$ all mutually perpendicular, and with $\vec{E} \times \vec{B}$ positive parallel to $\hat{r}$. Also note that $|\vec{E}| = c|\vec{B}|$. At very large distances from the oscillating dipole, the electromagnetic fields begin to resemble plane waves.

7.5 Dipole radiation (energy)

Electric and magnetic fields due to an oscillating dipole (assuming $z_m \ll r$ and $z_m \ll c/\omega = 1/k$):

$$\begin{aligned}
\vec{E} &= -\frac{\omega^2 p_m}{4\pi\epsilon_0 c^2 r}(\sin\theta(1 + i\eta - \eta^2)\hat{\theta} + 2\cos\theta(i\eta - \eta^2)\hat{r})e^{i(kr-\omega t)} \\
\vec{B} &= -\frac{\omega^2 p_m}{4\pi\epsilon_0 c^3 r}(\sin\theta(1 + i\eta)\hat{\phi})e^{i(kr-\omega t)}
\end{aligned}$$

where $\eta = c/(r\omega) = 1/(kr)$.

Far field ($r \gg c/\omega = 1/k$) implies $\eta \ll 1$, and so $\vec{E}$ and $\vec{B}$ (real part):

$$\begin{aligned}
\vec{E} &= -\frac{\omega^2 p_m \sin\theta}{4\pi\epsilon_0 c^2 r}\hat{\theta}\cos(kr - \omega t) \\
\vec{B} &= -\frac{\omega^2 p_m \sin\theta}{4\pi\epsilon_0 c^3 r}\hat{\phi})\cos(kr - \omega t)
\end{aligned}$$

It follows that

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \\
&= \frac{1}{\mu_0}\frac{\omega^4 p_m^2 \sin^2\theta}{(4\pi\epsilon_0)^2 c^5 r^2}\hat{\theta} \times \hat{\phi}\cos^2(kr - \omega t) \\
&= \frac{\omega^4 p_m^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3 r^2}\hat{r}\cos^2(kr - \omega t)
\end{aligned}$$

Time averaging yields (note: $\langle \cos^2(kr-\omega t)\rangle = 1/2$):

$$\langle \vec{S} \rangle = \frac{\omega^4 p_m^2 \sin^2\theta}{32\pi^2\epsilon_0 c^3 r^2}\hat{r} \tag{7.9}$$

Total power at distance r (integrated over all directions):

$$\begin{aligned}
\langle P \rangle &= \oint \langle \vec{S} \rangle \cdot d\vec{A} \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, \frac{\omega^4 p_m^2 \sin^2\theta}{32\pi^2\epsilon_0 c^3 r^2} \\
&= (2\pi)\frac{\omega^4 p_m^2}{32\pi^2\epsilon_0 c^3}\int_0^{\pi} d\theta\, \sin^3\theta \\
&= \frac{\omega^4 p_m^2}{16\pi\epsilon_0 c^3}\int_1^{-1}(-du)\,(1-u^2) \qquad (u=\cos\theta,\ du=-\sin\theta\, d\theta) \\
&= \frac{\omega^4 p_m^2}{16\pi\epsilon_0 c^3}(u-u^3/3)|_{-1}^{1} \\
&= \frac{\omega^4 p_m^2}{16\pi\epsilon_0 c^3}(2/3-(-2/3)) \\
&= \frac{\omega^4 p_m^2}{12\pi\epsilon_0 c^3}
\end{aligned}$$

It follows that

$$\langle \vec{S} \rangle = \frac{3}{2}\frac{\langle P \rangle}{4\pi r^2}\sin^2\theta \tag{7.10}$$

$$\langle P \rangle = \frac{\omega^4 p_m^2}{12\pi\epsilon_0 c^3} \tag{7.11}$$

Notice that radiation is not emitted isotropically, but instead is preferentially emitted along the plane perpendicular to the dipole (the x-y plane, where $\theta = \pi/2$), and is not emitted at all in the direction of the dipole oscillation (along $\pm z$ axis, where $\theta = 0$ or $\theta = \pi$). Also note that the total power emitted is proportional to ω^4 (the preferential scattering of higher frequency light is often used to explain blue skies and red sunrises/sunsets) and p_m^2, and is independent of r (consistent with energy conservation).

What is the energy current density in the regime where $z_m \ll r \lesssim 1/k$? In this case, η is not negligible and cannot be dropped. We first write $\vec{E}$ and $\vec{B}$ in non-phasor form, with the help of

$$\begin{aligned}
\mathrm{Re}(e^{i(kr-\omega t)}) &= \cos(kr-\omega t) = C \\
\mathrm{Re}(ie^{i(kr-\omega t)}) &= -\sin(kr-\omega t) = -S
\end{aligned}$$

where we abbreviate $\cos(kr-\omega t)$ and $\sin(kr-\omega t)$ by C and S, respectively. This yields

$$\begin{aligned}
\vec{E} &= -\frac{\omega^2 p_m}{4\pi\epsilon_0 c^2 r}(\sin\theta(C - S\eta - C\eta^2)\hat{\theta} + 2\cos\theta(-S\eta - C\eta^2)\hat{r}) \\
\vec{B} &= -\frac{\omega^2 p_m}{4\pi\epsilon_0 c^3 r}(\sin\theta(C - S\eta)\hat{\phi})
\end{aligned}$$

We will also define $C' = \cos 2(kr-\omega t)$ and $S' = \sin 2(kr-\omega t)$, and note that

$$\begin{aligned}
C^2 - S^2 &= C' \\
2CS &= S' \\
C^2 &= (1+C')/2
\end{aligned}$$

Putting this together yields

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} \\
&= \frac{1}{\mu_0}\frac{\omega^4 p_m^2}{(4\pi\epsilon_0)^2 c^5 r^2}(\sin\theta(C - S\eta - C\eta^2)\hat{\theta} + 2\cos\theta(-S\eta - C\eta^2)\hat{r}) \times (\sin\theta(C - S\eta)\hat{\phi}) \\
&= \frac{\omega^4 p_m^2}{16\pi^2\epsilon_0 c^3 r^2}(\sin^2\theta(\hat{\theta}\times\hat{\phi})(C^2 - 2CS\eta + S^2\eta^2 - C^2\eta^2 - CS\eta^3) \\
&\quad +2\sin\theta\cos\theta(\hat{r}\times\hat{\phi})(S^2\eta^2 - C^2\eta^2 - CS\eta + CS\eta^3)) \\
&= \frac{\omega^4 p_m^2}{32\pi^2\epsilon_0 c^3 r^2}(\sin^2\theta\hat{r}(1 + C' - S'(2\eta+\eta^3) - 2C'\eta^2) \\
&\quad +2\sin\theta\cos\theta(-\hat{\theta})(-2C'\eta^2 - S'(\eta-\eta^3)))
\end{aligned}$$

Given that $\langle S'\rangle = \langle C'\rangle = 0$, it is clear that $\langle\vec{S}\rangle$ is given by Eq. 7.9 (the result for $r \gg 1/k$) even if r is comparable to $1/k$, but that there may be significant time variations of this energy current within a given oscillation period in both the $\pm\hat{r}$ and $\pm\hat{\theta}$ directions. All of the time variations in the $\pm\hat{\theta}$ direction and much of the complexity in time variations in the $\pm\hat{r}$ direction disappear as the field point is moved very far away from the dipole oscillator ($\eta \to 0$).

A Appendix: Vector Calculus

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A vector field $\vec{A}$ can be written as the gradiant of a scalar field ($\vec{A} = \vec{\nabla}\phi$ for some ϕ) if and only if $\vec{\nabla}\times\vec{A} = 0$ at all points in space.

A vector field $\vec{B}$ can be written as the curl of a vector field ($\vec{B} = \vec{\nabla}\times\vec{A}$ for some $\vec{A}$) if and only if $\vec{\nabla}\cdot\vec{B} = 0$ at all points in space.

The "only if" parts of these two results are a simple matter of calculation:

$$\vec{\nabla}\times\vec{\nabla}\phi = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\begin{aligned}
(\vec{\nabla} \times \vec{\nabla}\phi)_x &= \partial_y(\vec{\nabla}\phi)_z - \partial_z(\vec{\nabla}\phi)_y \\
&= \partial_y\partial_z\phi - \partial_z\partial_y\phi \\
&= 0
\end{aligned}$$

Other components are similar...

$$\begin{aligned}
\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= \partial_x(\vec{\nabla} \times \vec{A})_x + \partial_y(\vec{\nabla} \times \vec{A})_y + \partial_z(\vec{\nabla} \times \vec{A})_z \\
&= \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) + \partial_z(\partial_x A_y - \partial_y A_x) \\
&= (\partial_y\partial_z A_x - \partial_z\partial_y A_x) + (\partial_z\partial_x A_y - \partial_x\partial_z A_y) + (\partial_x\partial_y A_z - \partial_y\partial_x A_z) \\
&= 0
\end{aligned}$$

To help with the proofs of the "if" parts, we will define $\bar{\partial}_x$, $\bar{\partial}_y$, and $\bar{\partial}_z$ to be the anti-derivative operators; specifically

$$\begin{aligned}
\bar{\partial}_x\psi(x,y,z) &= \int_0^x \psi(x',y,z)\,dx' \\
\bar{\partial}_y\psi(x,y,z) &= \int_0^y \psi(x,y',z)\,dy' \\
\bar{\partial}_z\psi(x,y,z) &= \int_0^z \psi(x,y,z')\,dz'
\end{aligned}$$

The following rules are satisfied:

$$\begin{aligned}
\partial_x\bar{\partial}_x\psi(x,y,z) &= \psi(x,y,z) \\
\bar{\partial}_x\partial_x\psi(x,y,z) &= \psi(x,y,z) - \psi(0,y,z) \\
\partial_y\bar{\partial}_x\psi(x,y,z) &= \bar{\partial}_x\partial_y\psi(x,y,z)
\end{aligned}$$

with similar rules involving other indices. Indeed,

$$\begin{aligned}
\partial_x\bar{\partial}_x\psi(x,y,z) &= \partial_x\int_0^x \psi(x',y,z)\,dx' \\
&= \psi(x,y,z)
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_x\partial_x\psi(x,y,z) &= \int_0^x \partial_{x'}\psi(x',y,z)\,dx' \\
&= \psi(x',y,z)|_0^x \\
&= \psi(x,y,z) - \psi(0,y,z)
\end{aligned}$$

$$\begin{aligned}\partial_y\bar{\partial}_x\psi(x,y,z) &= \partial_y\int_0^x \psi(x',y,z)\,dx' \\ &= \int_0^x \partial_y\psi(x',y,z)\,dx' \\ &= \bar{\partial}_x\partial_y\psi(x,y,z)\end{aligned}$$

To establish the first "if", we assume that $\vec{\nabla}\times\vec{A}=0$ and define

$$\phi(x,y,z) = \bar{\partial}_x A_x(x,0,0) + \bar{\partial}_y A_y(x,y,0) + \bar{\partial}_z A_z(x,y,z)$$

This is a concise way of writing $\phi(x,y,z)$ as the line integral of $\vec{A}$ from $(0,0,0)$ to (x,y,z) along the path

$$(0,0,0)\rightarrow(x,0,0)\rightarrow(x,y,0)\rightarrow(x,y,z)$$

Based on Stoke's Theorem, the path shouldn't matter (and it doesn't), but we need to provide a precise definition of ϕ.

It should be noted that $\vec{\nabla}\times\vec{A}=0$ is equivalent to $\partial_x A_y = \partial_y A_x$ and other similar identities. We then have

$$\begin{aligned}\partial_z\phi &= \partial_z\bar{\partial}_x A_x(x,0,0) + \partial_z\bar{\partial}_y A_y(x,y,0) + \partial_z\bar{\partial}_z A_z(x,y,z) \\ &= 0 + 0 + \partial_z\bar{\partial}_z A_z(x,y,z) \\ &= A_z(x,y,z)\end{aligned}$$

$$\begin{aligned}\partial_y\phi &= \partial_y\bar{\partial}_x A_x(x,0,0) + \partial_y\bar{\partial}_y A_y(x,y,0) + \partial_y\bar{\partial}_z A_z(x,y,z) \\ &= 0 + A_y(x,y,0) + \partial_y\bar{\partial}_z A_z(x,y,z) \\ &= A_y(x,y,0) + \bar{\partial}_z\partial_y A_z(x,y,z) \\ &= A_y(x,y,0) + \bar{\partial}_z\partial_z A_y(x,y,z) \\ &= A_y(x,y,0) + (A_y(x,y,z) - A_y(x,y,0)) \\ &= A_y(x,y,z)\end{aligned}$$

$$\begin{aligned}\partial_x\phi &= \partial_x\bar{\partial}_x A_x(x,0,0) + \partial_x\bar{\partial}_y A_y(x,y,0) + \partial_x\bar{\partial}_z A_z(x,y,z) \\ &= A_x(x,0,0) + \bar{\partial}_y\partial_x A_y(x,y,0) + \bar{\partial}_z\partial_x A_z(x,y,z) \\ &= A_x(x,0,0) + \bar{\partial}_y\partial_y A_x(x,y,0) + \bar{\partial}_z\partial_z A_x(x,y,z) \\ &= A_x(x,0,0) + (A_x(x,y,0) - A_x(x,0,0)) + (A_x(x,y,z) - A(x,y,0)) \\ &= A_x(x,y,z)\end{aligned}$$

It follows that $\vec{A}=\vec{\nabla}\phi$, completing the proof.

To establish the second "if", we assume that $\vec{\nabla}\cdot\vec{B}=0$ and define $\vec{A}$ by

$$A_x(x,y,z) = \bar{\partial}_z B_y(x,y,z) \qquad A_y(x,y,z) = -\bar{\partial}_z B_x(x,y,z) + \bar{\partial}_x B_z(x,y,0) \qquad A_z(x,y,z) = 0$$

It then follows that

$$\begin{aligned}
(\vec{\nabla} \times \vec{A})_x &= \partial_y A_z - \partial_z A_y \\
&= 0 - \partial_z(-\bar{\partial}_z B_x(x,y,z) + \bar{\partial}_x B_z(x,y,0)) \\
&= \partial_z \bar{\partial}_z B_x(x,y,z) - \partial_z \bar{\partial}_x B_z(x,y,0) \\
&= B_x(x,y,z) - 0 \\
&= B_x(x,y,z)
\end{aligned}$$

$$\begin{aligned}
(\vec{\nabla} \times \vec{A})_y &= \partial_z A_x - \partial_x A_z \\
&= \partial_z \bar{\partial}_z B_y(x,y,z) - 0 \\
&= B_y(x,y,z)
\end{aligned}$$

$$\begin{aligned}
(\vec{\nabla} \times \vec{A})_z &= \partial_x A_y - \partial_y A_x \\
&= \partial_x(-\bar{\partial}_z B_x(x,y,z) + \bar{\partial}_x B_z(x,y,0)) - \partial_y \bar{\partial}_z B_y(x,y,z) \\
&= -\partial_x \bar{\partial}_z B_x(x,y,z) + \partial_x \bar{\partial}_x B_z(x,y,0) - \partial_y \bar{\partial}_z B_y(x,y,z) \\
&= -\bar{\partial}_z \partial_x B_x(x,y,z) + B_z(x,y,0) - \bar{\partial}_z \partial_y B_y(x,y,z) \\
&= \bar{\partial}_z(-\partial_x B_x(x,y,z) - \partial_y B_y(x,y,z)) + B_z(x,y,0) \\
&= \bar{\partial}_z \partial_z B_z(x,y,z) + B_z(x,y,0) \qquad (\text{since } \vec{\nabla} \cdot \vec{B} = 0) \\
&= B_z(x,y,z) - B_z(x,y,0) + B_z(x,y,0) \\
&= B_z(x,y,z)
\end{aligned}$$

It follows that $\vec{B} = \vec{\nabla} \times \vec{A}$, completing the proof.

B Appendix: Complex Phasors

C Appendix: Delta Functions

1-D delta function ($\delta(x)$):

- $\delta(x) = 0$ if $x \neq 0$.
- $\delta(x)$ spikes to ∞ at $x = 0$.
- $\int_a^b \delta(x)\, dx = 1$ as long as $a < 0 < b$.
- More generally, $\int_a^b f(x)\delta(x - x_0)\, dx = f(x_0)$ as long as $a < x_0 < b$.

3-D delta function ($\delta(\vec{r})$):

- $\delta(\vec{r}) = 0$ if $\vec{r} \neq 0$.

- $\delta(\vec{r})$ spikes to ∞ at $\vec{r} = 0$.

- $\int_\Omega \delta(\vec{r})\, dV = 1$ as long as 0 is located in the interior of Ω (there exists an ϵ-ball around 0 entirely included in Ω).

- More generally, $\int_\Omega f(\vec{r})\delta(\vec{r} - \vec{r}_0)\, dV = f(\vec{r}_0)$ as long as $\vec{r}_0$ is located in the interior of Ω.

- It should be noted that $\delta(\vec{r})$ is a *scalar* even though $\vec{r}$ itself is a vector.

So our next question is, what is $\int_a^b f(x)\delta(g(x))\, dx$ (1-D) or $\int_\Omega f(\vec{r})\delta(\vec{g}(\vec{r}))\, dV$ (3-D)? Let's begin with 1-D and make the following assumptions:

- There is a single point x_0 where $g(x_0) = 0$, which lies in the interior of the interval $[a, b]$ (i.e., $a < x_0 < b$).

- The derivative of g at x_0 is not zero ($g'(x_0) \neq 0$).

Since $g(x_0) = 0$. it follows that $\delta(g(x))$ will spike near $x = x_0$. It should be noted that if $g'(x_0)$ were zero, that spike would be quite broad, and integrals involving $\delta(g(x))$ would have a tendency to diverge. That's why we are ruling that case out.

Anyway, since $\delta(g(x))$ spikes near x_0, it makes sense to expand $g(x)$ in a Taylor series around $x = x_0$:

$$\begin{aligned} g(x_0 + h) &= g(x_0) + g'(x_0)h + (1/2)g''(x_0)h^2 + \ldots \\ &= g_0'h + (1/2)g_0''h^2 + \ldots \end{aligned}$$

where we are using g_0', g_0'', etc., as shorthand for g' or g'' evaluated at x_0. Note that $g_0 = g(x_0) = 0$, so there is no need to include it.

Since the delta-function falls off quickly to zero as its argument moves away from zero, it makes sense to truncate the series after first order. It follows that

$$\delta(g(x)) = \delta(g_0'(x - x_0))$$

Note that this remains true even as $x - x_0$ grows sufficiently large that $g(x)$ is poorly approximated by $g_0'(x - x_0)$. What matters is that both $g(x)$ and $g_0'(x - x_0)$ are not zero, and so the delta functions on both sides will be zero.

To evaluate the integral

$$I = \int_a^b f(x)\delta(g(x))\, dx = \int_a^b f(x)\delta(g_0'(x - x_0))\, dx$$

we will make the substitution $u = |g_0'|(x - x_0)$ ($du = |g_0'|\, dx$). Since u is an increasing function of x, it follows that

$$u_a = |g_0'|(a - x_0) < 0 < |g_0'|(b - x_0) = u_b$$

and so the derivation proceeds

$$\begin{aligned}
I &= \int_a^b f(x)\delta(g_0'(x-x_0))\,dx \\
&= \int_{u_a}^{u_b} f(x_0+u/|g_0'|)\delta(\pm u)\,du/|g_0'| \\
&= (1/|g_0'|)\int_{u_a}^{u_b} f(x_0+u/|g_0'|)\delta(u)\,du \\
&= (1/|g_0'|)f(x_0) \\
&= f(x_0)/|g'(x_0)|
\end{aligned}$$

$$\int_a^b f(x)\delta(g(x))\,dx = f(x_0)/|g'(x_0)| \qquad (g(x_0)=0) \tag{C.1}$$

This should make sense. The closer $g'(x_0)$ is to zero, the broader the spike in $\delta(g(x))$ (x needs to differ more from x_0 in order that $g(x)$ differ appreciably from 0), and so the larger the integral will be. If $g'(x_0)$ were equal to zero, the spike could be quite broad, and the integral will surely diverge if $f(x_0)$ were not zero.

Now let's generalize to 3-D and consider the integral $\int_\Omega f(\vec{r})\delta(\vec{g}(\vec{r}))\,dV$. Again, we make the following assumptions:

- There is a single point $\vec{r}_0$ where $\vec{g}(\vec{r}_0)=0$, which lies in the interior of Ω.
- The gradient of $\vec{g}$ at $\vec{r}_0$ (represented by a matrix $\Gamma_{ij}=(\vec{\nabla}\vec{g})_{ij}=\partial_j g_i$) is non-singular.

Since $\vec{g}(\vec{r}_0)=0$, it follows that $\delta(\vec{g}(\vec{r}))$ will spike near $\vec{r}=\vec{r}_0$. Just as in the 1-D case, if $\Gamma=\vec{\nabla}\vec{g}(\vec{r}_0)$ were singular, that spike would be quite broad, especially along any direction belonging to the null-space of Γ, and integrals involving $\delta(\vec{g}(\vec{r}))$ would have a tendency to diverge.

Just as in the 1-D case, it makes sense to expand $\vec{g}(\vec{r})$ in a Taylor series around $\vec{r}=\vec{r}_0$:

$$\begin{aligned}
g_i(\vec{r}_0+\vec{h}) &= g_i(\vec{r}_0)+\sum_j h_j\partial_j g_i|_{\vec{r}_0}+(1/2)\sum_{j,k}\partial_j\partial_k g_i|_{\vec{r}_0}+\ldots \\
&= 0+\sum_j \Gamma_{ij}h_j+\ldots \qquad \text{(first order only)} \\
&= (\Gamma\vec{h})_i+\ldots
\end{aligned}$$

and so

$$\delta(\vec{g}(\vec{r}))=\delta(\Gamma(\vec{r}-\vec{r}_0))$$

Again, just as in the 1-D case, this latter relationship remains true even as $\vec{r}-\vec{r}_0$ grows sufficiently large that $\vec{g}(\vec{r})$ is poorly approximated by its first order expansion $\Gamma(\vec{r}-\vec{r}_0)$.

To evaluate the integral

$$I = \int_\Omega f(\vec{r})\delta(\vec{g}(\vec{r}))\,dV = \int_\Omega f(\vec{r})\delta(\Gamma(\vec{r}-\vec{r}_0))\,dV$$

we will make the substitution $\vec{u} = \Gamma(\vec{r}-\vec{r}_0)$. Since Γ is linear and non-singular, 0 will lie in the interior of $\Omega' = \Gamma_>(\Omega)$ and $dV_{\vec{u}} = |\det\Gamma|\,dV_{\vec{r}}$. The derivation proceeds

$$\begin{aligned}
I &= \int_\Omega f(\vec{r})\delta(\Gamma(\vec{r}-\vec{r}_0))\,dV_{\vec{r}} \\
&= \int_{\Omega'} f(\vec{r}_0+\Gamma^{-1}\vec{u})\delta(\vec{u})\,dV_{\vec{u}}|\det\Gamma|^{-1} \\
&= |\det\Gamma|^{-1}\int_{\Omega'} f(\vec{r}_0+\Gamma^{-1}\vec{u})\delta(\vec{u})\,dV_{\vec{u}} \\
&= |\det\Gamma|^{-1} f(\vec{r}_0+\Gamma^{-1}(0)) \\
&= f(\vec{r}_0)/|\det\Gamma|
\end{aligned}$$

$$\int_\Omega f(\vec{r})\delta(\vec{g}(\vec{r}))\,dV = f(\vec{r}_0)/|\det\Gamma| \qquad (\vec{g}(\vec{r}_0)=0,\ \Gamma_{ij} = \partial_j g_i|_{\vec{r}_0}) \tag{C.2}$$

The spike region of $\delta(\vec{g}(\vec{r}))$ is the image of the spike region of $\delta(\vec{r})$ under the mapping Γ^{-1}. It makes sense that its volume is adjusted by $1/(\det\Gamma)$, which directly impacts the integral.

D Appendix: The Wave Equation

D.1 Laplacian of $1/r$

Our goal in this section is to evaluate $\nabla^2(1/r)$ where $r = |\vec{r}|$.

We first note that

$$\begin{aligned}
r\partial_i r &= \partial_i(\frac{1}{2}r^2) \\
&= \partial_i(\frac{1}{2}\vec{r}\cdot\vec{r}) \\
&= \vec{r}\cdot\partial_i(\vec{r}) \\
&= \vec{r}\cdot\partial_i(\sum_j x_j\hat{x}_j) \\
&= \vec{r}\cdot\hat{x}_i \\
&= x_i
\end{aligned}$$

It follows that $\partial_i r = x_i/r$ and so

$$\vec{\nabla} r = \vec{r}/r = \hat{r}$$

We thus find that

$$\begin{aligned}
\vec{\gamma} &= \vec{\nabla}(1/r) \\
&= \vec{\nabla}(r)\frac{d(1/r)}{dr} \qquad \text{(chain rule)} \\
&= -\hat{r}/r^2 \\
&= -\vec{r}/r^3
\end{aligned}$$

The Laplacian is then given by

$$\begin{aligned}
\nabla^2(1/r) &= \vec{\nabla}\cdot\vec{\gamma} \\
&= -(\vec{\nabla}\cdot\vec{r})/r^3 - (\vec{r}\cdot\vec{\nabla}(r^{-3})) \\
&= -(\sum_j \partial_j x_j)/r^3 - \vec{r}\cdot(\hat{r})(-3r^{-4}) \\
&= -(\sum_j 1)/r^3 + r(3/r^4)p \\
&= -3/r^3 + 3/r^3 \\
&= 0
\end{aligned}$$

This derivation is certainly valid for $\vec{r} \neq 0$. But what happens at $\vec{r} = 0$? To help answer this question, let's integrate $\nabla^2(1/r)$ over a spherical ball of radius R centered at the origin. On the one hand, we might expect to get zero, since $\nabla^2(1/r)$ is zero at all points except possibly at the origin (and generally speaking, values of a function at a single point do not affect integrals, even if the integration region includes that point). However, with the help of the divergence theorem, we find that

$$\begin{aligned}
\int_{\mathrm{sph}(R)} \nabla^2(1/r)\,dV &= \int_{\mathrm{sph}(R)} \vec{\nabla}\cdot\vec{\gamma}\,dV \\
&= \oint_{\partial(\mathrm{sph}(R))} \vec{\gamma}\cdot d\vec{A} \\
&= \oint_{\partial(\mathrm{sph}(R))} (-1/r^2)\hat{r}\cdot(dA\hat{r}) \\
&= \oint_{\partial(\mathrm{sph}(R))} (-1/r^2)\,dA \\
&= (-1/R^2)\oint_{\partial(\mathrm{sph}(R))} dA \\
&= (-1/R^2)(4\pi R^2) \\
&= -4\pi
\end{aligned}$$

Last time I checked, that value isn't zero.

This implies that

$$\nabla^2(1/r) = -4\pi\delta(\vec{r}) \tag{D.1}$$

where $\delta(\vec{r})$ represents the 3-dimensional delta-function.

The (3-dimensional) delta function represents the limit of a series of functions which spike near $\vec{r} = 0$, and is characterized by the following conditions:

- $\delta(\vec{r}) = 0$ whenever $\vec{r} \neq 0$.
- $\int_\Omega \delta(\vec{r})\, dV = 1$ (and so, more generally, $\int_\Omega \delta(\vec{r}) f(\vec{r})\, dV = f(0)$) whenever the origin belongs to the interior of the region Ω.

Of course no such function can satisfy both conditions above at the same time. However, one can easily construct a series of functions (based on a small parameter ϵ) that are zero (or very nearly zero) for $|\vec{r}| > \epsilon$ and spike in value when $|\vec{r}| < \epsilon$ such that the integral of that function will (nearly) equal 1 whenever the origin belongs to the given integration region and is at least ϵ away from the region's boundary. One can then treat $\delta(\vec{r})$ as the limit of such functions as $\epsilon \to 0$.

It should also be noted that if we integrate $\nabla^2(1/r)$ over a spherical shell with inner radius R_1 and outer radius R_2 (with $R_2 > R_1 > 0$), then the result *will* be zero:

$$\begin{aligned}
\int_{\text{shell}(R_1,R_2)} \nabla^2(1/r)\, dV &= \int_{\text{shell}(R_1,R_2)} \vec{\nabla} \cdot \vec{\gamma}\, dV \\
&= \oint_{\partial(\text{shell}(R_1,R_2))} \vec{\gamma} \cdot d\vec{A} \\
&= \int_{\text{surf}(R_1)} (-1/r^2)\hat{r} \cdot (dA(-\hat{r})) + \int_{\text{surf}(R_2)} (-1/r^2)\hat{r} \cdot (dA(+\hat{r})) \\
&= (-1/R_1^2)(-4\pi R_1^2) + (-1/R_2^2)(+4\pi R_2^2) \\
&= (+4\pi) + (-4\pi) \\
&= 0
\end{aligned}$$

This result remains true as the inner radius R_1 approaches zero, since the singularity at the origin remains excluded from the integration volume, even for small (non-zero) values of R_1. It is clear that the non-zero integral over the spherical ball is entirely due to the singularity of $\nabla^2(1/r)$ at the origin.

D.2 Generalization to $f(r)/r$

We will now consider a generalization of last section, and calculate the Laplacian of

$$\psi(\vec{r}) = f(r)/r \qquad (r = |\vec{r}|)$$

where $f(r)$ is an arbitrary (twice-differentiable) function (first and second derivatives written $f'(r)$ and $f''(r)$, respectively).

$$
\begin{aligned}
\vec{\gamma} &= \vec{\nabla}\psi \\
&= \vec{\nabla}(r)(d/dr)(f(r)/r) \\
&= (\hat{r})(f'(r)/r + f(r)(-1/r^2)) \\
&= (f'(r)/r - f(r)/r^2)\hat{r} \\
&= (f'(r)/r^2 - f(r)/r^3)\vec{r}
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2\psi &= \vec{\nabla}\cdot\vec{\gamma} \\
&= \vec{\nabla}(f'(r)r^{-2} - f(r)r^{-3})\cdot\vec{r} + (f'(r)r^{-2} - f(r)r^{-3})(\vec{\nabla}\cdot\vec{r}) \\
&= (f''(r)r^{-2} + f'(r)(-2r^{-3}) - f'(r)r^{-3} - f(r)(-3r^{-4}))(\hat{r}\cdot\vec{r}) + (f'(r)r^{-2} - f(r)r^{-3})(3) \\
&= f''(r)r^{-1} - 3f'(r)r^{-2} + 3f(r)r^{-3} + 3f'(r)r^{-2} - 3f(r)r^{-3} \\
&= f''(r)/r
\end{aligned}
$$

This is certainly true when $\vec{r} \neq 0$. As you might guess, in order to properly handle integrals of $\nabla^2\psi$ over regions which include the origin as an interior point, it will be necessary to add a delta-function term. Thus,

$$\nabla^2\psi = f''(r)/r + C\delta(\vec{r})$$

where C is a constant we will be determining shortly.

On the one hand,

$$
\begin{aligned}
\int_{\mathrm{sph}_R} \nabla^2\psi\,dV &= \int_{\mathrm{sph}_R} (f''(r)/r + C\delta(\vec{r}))\,dV \\
&= C + \int_0^R dr \int_0^\pi r\,d\theta \int_0^{2\pi} r\sin\theta\,d\phi\, f''(r)/r \\
&= C + 4\pi\int_0^R r^2(f''(r)/r)\,dr \\
&= C + 4\pi\int_0^R rf''(r)\,dr \\
&= C + 4\pi(rf'(r)|_0^R - \int_0^R f'(r)\,dr) \qquad \text{(by parts: } u = r,\ dv = f''(r)\,dr) \\
&= C + 4\pi(Rf'(R) - (f(R) - f(0)))
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
\int_{\mathrm{sph}_R} \nabla^2\psi\, dV &= \int_{\mathrm{sph}_R} \vec{\nabla}\cdot\vec{\gamma}\, dV \\
&= \oint_{\partial(\mathrm{sph}_R)} \vec{\gamma}\cdot d\vec{A} \\
&= \oint_{\partial(\mathrm{sph}_R)} (f'(r)/r - f(r)/r^2)\hat{r}\cdot(dA\,\hat{r}) \\
&= (f'(R)/R - f(R)/R^2)\oint_{\partial(\mathrm{sph}_R)} dA \\
&= (f'(R)/R - f(R)/R^2)4\pi R^2 \\
&= 4\pi(Rf'(R) - f(R))
\end{aligned}$$

Setting these two equal to each other yields

$$C + 4\pi(Rf'(R) - f(R) + f(0)) = 4\pi(Rf'(R) - f(R))$$

which implies

$$C = -4\pi f(0)$$

Note that C is independent of R, as it should be.

It thus follows that

$$\nabla^2(f(r)/r) = f''(r)/r - 4\pi f(0)\delta(\vec{r}) \tag{D.2}$$

Note that Eq. D.2 reduces to Eq. D.1 when we set $f(r) = 1$.

D.3 Time dependence

We will now introduce some time dependence and consider the following field

$$\psi(\vec{r},t) = f(t - r/c)/r \qquad (r = |\vec{r}|)$$

If we fix t and define $g(r) = f(t - r/c)$, then $g'(r) = (-1/c)f'(t - r/c)$ and $g''(r) = (-1/c)^2 f''(t - r/c)$, and so

$$\nabla^2\psi = g''(r)/r - 4\pi g(0)\delta(\vec{r}) = (1/c^2)f''(t - r/c)/r - 4\pi f(t)\delta(\vec{r})$$

We can also consider time derivatives (for fixed values of $\vec{r}$), and find that $\partial_t f(t - r/c) = f'(t - r/c)$ and $\partial_t^2 f(t - r/c) = f''(t - r/c)$. It follows that

$$\partial^2\psi/\partial t^2 = f''(t - r/c)/r$$

Putting these two facts together, we find that

$$\begin{aligned}
\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} &= (1/c^2)f''(t - r/c)/r - 4\pi f(t)\delta(\vec{r}) - (1/c^2)f''(t - r/c)/r \\
&= -4\pi f(t)\delta(\vec{r})
\end{aligned}$$

If we replace $\vec{r}$ by $\vec{r} - \vec{r}'$ (effectively shifting the origin to $\vec{r}'$ instead of 0), we obtain the following result:

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi f(t)\delta(\vec{r} - \vec{r}') \qquad (\psi(\vec{r}) = f(t - |\vec{r} - \vec{r}'|/c)/|\vec{r} - \vec{r}'|) \tag{D.3}$$

Note that the field ψ represents a spherical pulse eminating from $\vec{r}'$ and travelling outwards at a speed c in all directions. Eq. D.3 has significant implications regarding travelling waves produced by a point source, as we will be discussing in the next section.

D.4 Non-dispersive waves

Waves are normally described by providing a field $\psi(\vec{r}, t)$ representing "displacement from equilibrium". There may be "disturbances" which are present in regions of space where ψ differs significantly from zero, which can then travel to other regions of space as time passes.

The propagation of non-dispersive waves in particular (i.e., waves for which the wave speed is constant, independent of wavelength, etc) are often well-modeled by the following *source-free wave equation*:

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{D.4}$$

In this equation, c represents the (constant) speed of the wave.

Examples of such waves include waves in a string (inherently 1 dimensional), sound waves, and electromagnetic waves.

A "plane-wave" solution is given by

$$\psi(\vec{r}, t) = A\cos(\vec{k} \cdot \vec{r} - \omega t)$$

where A is the amplitude, $\vec{k}$ is the wave-vector (which can be related to the wave-length), ω is the angular frequency, and $\omega/|\vec{k}|$ is equal to the wave speed c.

Indeed, we have

$$\begin{aligned}
\partial_\mu \psi(\vec{r}, t) &= A(-\sin(\vec{k} \cdot \vec{r} - \omega t))\partial_\mu(\sum_j k_j x_j - \omega t) \\
&= \begin{cases} Ak_i(-\sin(\vec{k} \cdot \vec{r} - \omega t)) & \mu = i = x,\, y, \text{ or } z \\ A(-\omega)(-\sin(\vec{k} \cdot \vec{r} - \omega t)) & \mu = t \end{cases}
\end{aligned}$$

Applying ∂_μ again converts $-\sin$ into $-\cos$ and brings out another factor of k_i or $-\omega$ as the case may be. It then follows that

$$\begin{aligned}
\partial_i^2 \psi(\vec{r}, t) &= -k_i^2 \psi(\vec{r}, t) \\
\partial_t^2 \psi(\vec{r}, t) &= -(-\omega)^2 \psi(\vec{r}, t)
\end{aligned}$$

Putting all of this together, we get

$$\begin{aligned}
\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} &= -(\sum_i k_i^2)\psi - (-\omega^2/c^2)\psi \\
&= (\omega^2/c^2 - |\vec{k}|^2)\psi \\
&= (0)\psi \\
&= 0
\end{aligned}$$

This wave is called a plane wave because the wave-fronts, characterized by points satisfying

$$\vec{k}\cdot\vec{r} - \omega t = 2\pi n \qquad (n \text{ integer})$$

are planes perpendicular to $\vec{k}$. The planes then travel in the direction of $\vec{k}$ over time with speed equal to $\omega/|\vec{k}|$.

While the source-free wave equation, Eq. D.4, provides a good model for wave propagation, it does not explain how such waves get started. The plane wave solution in particular is a wave that exists everywhere and for all time. Presumably there is a far-away source that produced this wave quite some time ago, but it is not being modeled.

On the other hand, the production and subsequent propagation of a wave generated by a point source located at point $\vec{r}'$, and characterized by a time-dependent forcing function $f(t)$, is well-modeled by the differential equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = f(t)\delta(\vec{r} - \vec{r}') \tag{D.5}$$

According to Eq. D.3 from the last section, a solution to this equation is given by

$$\psi(\vec{r}, t) = -\frac{1}{4\pi}\frac{f(t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|}$$

Note the implication of this solution: the field at position $\vec{r}$ (which is a distance $|\vec{r} - \vec{r}'|$ away from the source) and time t is determined by the source at an earlier time $t - |\vec{r} - \vec{r}'|/c$. This time is often referred to as the "retarded time". Basically, the source emits a signal at the earlier time, and then that signal propagates outward in all directions with speed c, whereupon it reaches position $\vec{r}$ (after a travel time of $|\vec{r} - \vec{r}'|/c$) at time t.

If the forcing function is sinusoidal (i.e., $f(t) = A\cos(\omega t)$), then the emitted signal is an outward travelling spherical wave given by

$$\psi(\vec{r}, t) = -\frac{1}{4\pi}\frac{A\cos(\omega t - k|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|}$$

where $k = \omega/c$. Note that the amplitude decreases by a factor of one over distance as the wave spreads outwards — this turns out to be consistent with conservation of energy.

On the other hand, if the forcing function is a constant (i.e., $f(t) = K$, independent of time), then the resulting field will also be independent of time, and will be given by

$$\psi(\vec{r}, t) = -\frac{1}{4\pi}\frac{K}{|\vec{r} - \vec{r}\,'|}$$

Basically, this amounts to a $1/r$ field being generated by a persistent source. Note that the time delay between the source and the resulting field is still technically present, even though it is hidden by the fact that the source is time-independent. It is still the case that the field at position $\vec{r}$ and time t is based on what the source was doing at an earlier time — it's just that the source isn't changing so that is not clearly obvious from the formula.

If there are multiple sources, it is appropriate to use superposition. In the most general case, we can consider some arbitrary time-dependent source density function $\alpha(\vec{r}, t)$. The production and propagation of the wave from this source distribution is well modeled by the following differential equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \alpha(\vec{r}, t) \tag{D.6}$$

The general solution is given by

$$\psi(\vec{r}, t) = -\frac{1}{4\pi}\int \frac{\alpha(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c)}{|\vec{r} - \vec{r}\,'|}\, dV' \tag{D.7}$$

In Eq. D.7 we integrate $\vec{r}\,'$ over the entire 3-dimensional space.

Indeed,

$$\begin{aligned}
(\nabla^2 - (1/c^2)\partial_t^2)\psi &= -\frac{1}{4\pi}\int (\nabla^2 - (1/c^2)\partial_t^2)\left(\frac{\alpha(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c)}{|\vec{r} - \vec{r}\,'|}\right) dV' \\
&= -\frac{1}{4\pi}\int (-4\pi)\alpha(\vec{r}\,', t)\delta(\vec{r} - \vec{r}\,')\, dV' \\
&= -\frac{1}{4\pi}(-4\pi)\alpha(\vec{r}, t) \\
&= \alpha(\vec{r}, t)
\end{aligned}$$

Just to clarify, it should be noted that ∇^2 operates on $\vec{r}$, whereas $\vec{r}\,'$ is the integration variable and should be regarded as independent of $\vec{r}$. Thus, $\vec{r}\,'$ is a constant as far as ∇^2 (and of course ∂_t^2) is concerned. Accordingly, the first argument of α ($\vec{r}\,'$) is simply a parameter that "goes along for the ride".

E Appendix: Tensors

Hi there.